

# 第一章 多项式习题解答

$$\text{P44.1 } 1) f(x) = g(x)\left(\frac{1}{3}x - \frac{7}{9}\right) + \left(-\frac{26}{9}x - \frac{2}{9}\right)$$

$$2) f(x) = g(x)(x^2 + x - 1) + (-5x + 7)$$

$$\text{P44.2 } 1) x^2 + mx - 1 \mid x^3 + 9x + q \Rightarrow \text{余式 } (p+1+m^2)x + (q-m) = 0$$

$$\therefore \begin{cases} m = q \\ p = q^2 - 1 \end{cases}$$

方法二,

$$\text{设 } x^3 + px + q = (x^2 + m - 1)(x + q) \Rightarrow \begin{cases} m - q = 0 \\ -mq - 1 = p \end{cases} \text{ 同样。}$$

$$2) x^2 + mx + 1 \mid x^4 + px^2 + q \Rightarrow \text{余式 } m(p+2-m^2)x - (q-p+1+m^2) = 0$$

$$\therefore m(m^2 + p - 2) = 0. \quad m^2 + p = 1 + q, (x^2 = 1 - p + q)$$

$$\text{P44.3.1 用 } g(x) = x + 3 \text{ 除 } f(x) = 2x^5 - 5x^3 - 8x$$

解:

$$\therefore f(x) = 2(x+3)^5 - 30(x+3)^4 + 175(x+3)^3 - 495(x+3)^2 + 667(x+3) - 327$$

**P44.3.2)**

$$\begin{aligned} & \therefore (x^3 - x^2 - x) \\ &= (x-1+2i)^3 + (2-8i)(x-1+2i)^2 \\ & \quad - (12+8i)(x-1+2i) - (9-8i) \\ & \text{即余式 } -9+8i \\ & \text{商 } x^2 - 2ix - (5+2i) \end{aligned}$$

$$\text{P44.4.1. } f(x) = x^5, x_0 = 1: \text{ 即}$$

$$\therefore f(x) = (x-1)^5 + 5(x-1)^4 + 10(x-1)^3 + 10(x-1)^2 + 5(x-1) + 1$$

$$\text{当然也可以 } f(x) = x^5 = [(x-1)+1]^5$$

$$= (x-1)^5 + 5(x-1)^4 + \cdots + 1$$

**P44.4 2) 结果**

$$f(x) = x^4 - 2x^2 + 3 = (x+2)^4 - 8(x+2)^3 + 22(x+2)^2 - 24(x+2) + 11$$

**3)**

$$\begin{aligned} f(x) &= x^4 + 2ix^3 - (1+i)x^2 + 3x + 7 + i \\ &= (x+i-i)^4 + 2i(x+i-i)^3 - (1+i)(x+i-i)^2 - 3(x+i-i) + 7 + i \\ &= (x+i)^4 - 2i(x+i)^3 + (1+i)(x+i)^2 - 5(x+i) + 7 + 5i \end{aligned}$$

**P45.5**

$$(1) g(x) = (x-1)(x^2 + 2x + 1) = (x-1)(x+1)^2$$

$$f(x) = (x+1)(x^3 - 3x - 1)$$

$$\therefore (f(x), g(x)) = x+1$$

$$(2) g(x) = x^3 - 3x^2 + 1 \text{ 不可约}$$

$$f(x) = x^4 - 4x^3 + 1 \text{ 不可约}$$

$$\therefore (f(x), g(x)) = 1$$

$$(3) f(x) = x^4 - 10x^2 + 1 = (x^2 + 2\sqrt{2}x - 1)(x^2 - 2\sqrt{2}x - 1)$$

$$g(x) = x^4 - 4\sqrt{2}x^3 + 6x^2 + 6\sqrt{2}x + 1, f(x) = 4\sqrt{2}(-x^3 + 2\sqrt{2}x^2 + x) = (x^2 - 2\sqrt{2}x - 1)^2 \therefore$$

$$(f(x), g(x)) = x^2 - 2\sqrt{2}x - 1$$

#### P45.6

$$(1) f(x) = (x+1)^2(x^2-2) \quad g(x) = (x^2-2)(x^2+x+1)$$

$$\therefore (x+1)^2[-(x+1)] + (x^2+x+1)(x+2) = 1$$

$$\therefore (x^2-2) = -(x+1)f(x) + (x+2)g(x)$$

$$(2) f(x) = (x-1)(4x^3+2x^2-14x-y), \quad g(x) = (x-1)(2x^2+x-4)$$

$$= (x-1)f_1(x) \quad = (x-1)g_1(x)$$

而

$$f_1(x) = g_1(x) \cdot 2x - 3(2x+3)$$

$$g_1(x) = (2x+3) \cdot (x-1)$$

$$\therefore 1 = (2x+3)(x-1) - g_1 = \left(\frac{2x}{3}y_1 - \frac{1}{3}f_1\right)(x-1) - g_1$$

$$\therefore x-1 = -\frac{1}{3}(x-1)f(x) + \left(\frac{2}{3}x^2 - \frac{2}{3}x - 1\right)g(x)$$

$$(3) f(x) = x^4 - x^3 - 4x^2 + 4x + 1, \quad g(x) = x^2 - x - 1$$

$$\therefore f(x) = g(x)(x^2-3) + (x-2), \quad g(x) = (x-2)(x+1) + 1$$

$$\therefore 1 = -(f - g(x^2-3))(x+1) + g$$

$$= -(x+1)f(x) + (x^3+x^2-3x-2)g(x)$$

#### P45.7

$$f(x) = g(x)1 + (1+t)x^2 + (2-t)x + u = r(x)$$

$$g(x) = r(x)\left(\frac{1}{1+t}x + \frac{t-2}{(1+t)^2}\right) + \frac{(t^2+t+u)+(t-2)^2}{(1+t)^2}x + \left(1 - \frac{t-2}{(1+t)^2}\right)u$$

由题意  $r(x)$  与  $g(x)$  的公因式应为二次所以  $r(x) \mid g(x)$

$$\therefore \begin{cases} \frac{t^3 + 3t^2 - (u+3)t + (4-u)}{(1+t)^2} = 0 \\ \frac{t^2 + t + 3}{(1+t)^2}u = 0 \end{cases}$$

$t \neq -1$  否则  $r(x)$  为一次的

$$\therefore \Rightarrow \begin{cases} t^3 + 3t^2 - (u+3)t + (4-u) = 0 \\ (t^2 + t + 3)u = 0 \end{cases}$$

解出 (i) 当  $u=0$  时  $t^3 + 3t^2 - 3t + 4 = 0(t+4)(t^2 - t + 1)$

$$\therefore t = -4 \text{ 或 } t = \frac{1 \pm \sqrt{3}i}{2} = e^{\pm \frac{\pi}{3}i}$$

(ii) 当  $u \neq 0$  时, 只有  $t^2 + t + 3 = 0, \frac{1}{t+1} = -\frac{t}{3}$

$$t^3 + 3t^2 - (u+3)t + (4-u) \Rightarrow u = \frac{t^3 + 3t^2 - 3t + 4}{t+1} = -\frac{t}{3}(t^3 + 3t^2 - 3t + 4)$$

$$\therefore u = -\frac{1}{3}[(t^2 + t + 3)(t^2 + 2t - 8) + 6t + 24] = -2(t+4)$$

$$\text{即 } \begin{cases} u = -2(t+4) \\ t^2 + t + 3 = 0 \end{cases}$$

$$t = \frac{-1 \pm \sqrt{-11}}{2}$$

P45、8  $d(x) | f(x), d(x) | g(x)$  表明  $d(x)$  是公因式

又已知:  $d(x)$  是  $f(x)$  与  $g(x)$  的组合 表明任何公因式整除  $d(x)$

所以  $d(x)$  是一个最大的公因式。

P45、9. 证明  $(f(x)h(x), g(x)h(x)) = (f(x), g(x))h(x)$  ( $h(x)$  的首系=1)

证: 设  $(f(x)h(x), g(x)h(x)) = m(x)$  由

$$(f(x), g(x))h(x) | f(x)h(x) \quad (f(x), g(x))h(x) | g(x)h(x).$$

$$\therefore (f(x), g(x))h(x) | m(x) \quad \therefore (f(x), g(x))h(x) \text{ 是一个公因式.}$$

$$\text{设 } d(x) = (f(x), g(x)) = u(x)f(x) + v(x)g(x).$$

$$\therefore d(x)h(x) = (f(x), g(x))h(x) = u(x)f(x)h(x) + v(x)g(x)h(x).$$

而首项系数=1, 又是公因式得 (由 P45、8), 它是最大公因式, 且

$$(f(x), g(x))h(x) = (f(x)h(x), g(x)h(x)).$$

P45、10 已知  $f(x), g(x)$  不全为 0。证明  $(\frac{f(x)}{(f(x), g(x))}, \frac{g(x)}{(f(x), g(x))}) = 1$ .

证: 设  $d(x) = (f(x), g(x))$ . 则  $d(x) \neq 0$ .

$$\text{设 } \frac{f(x)}{d(x)} = f_1(x), \quad \frac{g(x)}{d(x)} = g_1(x), \quad \text{及 } d(x) = u(x)f(x) + v(x)g(x).$$

$$\text{所以 } d(x) = u(x)f_1(x)d(x) + v(x)g_1(x)d(x).$$

$$\text{消去 } d(x) \neq 0 \text{ 得 } 1 = u(x)f_1(x) + v(x)g_1(x)$$

**P45.11** 证: 设  $(f(x), g(x)) = d(x) \neq 0, f(x) = f_1(x)d(x), g(x) = g_1(x)d(x)$

$$\therefore u(x)f_1(x)d(x) + v(x)g_1(x)d(x) = d(x), u(x)f_1(x) + v(x)g_1(x) = 1$$

**P45.12**

设  $uf + vg = 1, u_1f + v_1h = 1 \Rightarrow uu_1f^2 + ufv_1h + vgu_1f + vu_1gh = 1$   
 $\therefore (uu_1f + uv_1h + vgu_1)f + (v_1u)gh = 1 \Rightarrow (f, gh) = 1$ .

#### P45.13

$$\therefore (f_i, g_i) = 1,$$

$$\text{固定 } i: (f_i, g_1g_2) = 1$$

$$(f_i, g_1 \cdot g_2 \cdot g_n) = 1$$

#### P45.14

$$(f, g) = 1 \Rightarrow uf + vg = 1 \Rightarrow (u - v)f + v(g + f) = 1 \Rightarrow (f, g + f) = 1$$

$$\text{同理 } (g, g + f) = 1.$$

$$\text{由 12 题 } (fg, f + g) = 1$$

$$\text{令 } g = g_1g_2 \cdots g_n$$

$$\therefore \text{每个 } i, (f_i, g) = 1$$

$$\Rightarrow (f_1f_1, g) = 1,$$

$$\Rightarrow (f_1f_2f_3, g) = 1,$$

$$\Rightarrow (f_1f_2 \cdots f_m, g_1g_2 \cdots g_n) = 1 \quad (\text{注反复归纳用 12 题}).$$

推广

若  $(f(x), g(x)) = 1$ , 则  $\forall m, n$  有  $(f(x)^m, g(x)^n) = 1$

P45.15

$$f(x) = x^3 + 2x^2 + 2x + 1, g(x) = x^4 + x^3 + 2x^2 + x + 1$$

$$\text{解: } g(x) = f(x)(x-1) + 2(x^2 + x + 1),$$

$$f(x) = (x^2 + x + 1)(x+1)$$

$$\text{即 } (f(x), g(x)) = x^2 + x + 1.$$

$$\text{令 } (x^2 + x + 1) = 0 \text{ 得 } \varepsilon_1 = \frac{-1 + \sqrt{3}i}{2}, \varepsilon_2 = \frac{-1 - \sqrt{3}i}{2}$$

$$\therefore f(x) \text{ 与 } g(x) \text{ 的公共根为 } \varepsilon_1, \varepsilon_2.$$

#### P45.16 判断有无重因式

$$\textcircled{1} f(x) = x^5 - 5x^4 + 7x^3 + 2x^2 + 4x - 8 \quad \textcircled{2} f(x) = x^4 + 4x^2 - 4x - 3$$

$$\text{解 } \textcircled{1} f'(x) = 5x^4 - 20x^3 + 21x^2 - 4x + 4$$

$$5f(x) = f'(x)(x-1) - 3(2x^3 - 5x^2 - 4x + 12)$$

$$f'(x) = (2x^3 - 5x^2 - 4x + 12)(5x - \frac{15}{2}) + \frac{49}{2}(x^2 - 4x + 4)$$

$$(2x^3 - 5x^2 - 4x + 12) = (x^2 - 4x + 4)(2x + 3)$$

故  $f(x)$  有重因式  $(x-2)^3$

$$\textcircled{2} f'(x) = 4x^3 + 8x - 4$$

$$f(x) = (x^3 + 2x - 1)x + (2x^2 - 3x + 3)$$

$$f'(x) = (2x^2 - 3x + 3)(2x + 3) + (11x - 13)$$

$$11(2x^2 - 3x + 3) = (11x - 13)(2x - \frac{6}{11}) + (33 + \frac{6 \times 13}{11})$$

$$\therefore (f(x) \cdot f'(x)) = 1$$

**P45.17**  $t = ?$  时  $f(x) = x^3 - 3x^2 + tx - 1$  有重因式 (有重根)

$$\text{解. } f'(x) = 3x^2 - 6x + t \quad 3f(x) = f'(x)(x-1) + (2t-6)x + (t-3)$$

$$\text{如 } t = 3 \text{ 则 有重因式: } 3 \text{ 重因式 } (x-1)^3 = f(x)$$

$$\text{如 } t \neq 3 \text{ 则 } 2f'(x) = (2x+2)(3x - \frac{15}{2}) + (2t + \frac{15}{2})$$

$$\text{此时必须 } t = -\frac{15}{4} \text{ 有重因式 } f(x) = (x + \frac{1}{2})^2(x-4)$$

**P45.18** 求多项式  $f(x) = x^3 + px + q$  有重根因式的条件

$$\text{证 } f(x) = 3x^2 + p$$

$$3f(x) = (3x^2 + p)x + 2px + 3q \quad (p \neq 0)$$

$$(3x^2 + p) = (2px + 3q)(\frac{3}{2p}x - \frac{3a}{4p^2}) + (p + \frac{27q^2}{4p^2})$$

$$\therefore 4p^3 + 27q^2 = 0$$

**p45.19** 令  $f(x) = Ax^4 + Bx^2 + 1$ , 因为  $(x-1)^2 \mid f(x)$ , 所以  $(x-1) \mid f'(x)$

$$\text{即 } f'(x) = 4Ax^3 + 2Bx = (x-1)(ax^2 + bx + c)$$

$$\begin{cases} a = 4A \\ b - a = 0 \\ c - b = 2B \\ -c = 0 \end{cases} \quad \begin{aligned} &\therefore 4A = a = b = -2B \\ &\therefore 2A + B = 0 \end{aligned}$$

$$\text{又 } (x-1)f(x) \Rightarrow f(x) = (x-1)(a'x^3 + b'x^2 + c'x + d')$$

$$\begin{cases} a' = A \\ b' - a' = 0 \\ c' - b' = 0 \\ -d' = 1 \end{cases} \quad \begin{aligned} &\therefore a' = A \\ &b' = a' \\ &-1 - b' = B \end{aligned} \quad \therefore A + B + 1 = 0$$

$$\therefore A = 1, B = -2$$

**P46, 20** 证  $f(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$  无重因式 (重根)

$$\text{证: } f'(x) = f(x) - \frac{x^n}{n!}$$

$$\therefore (f', f) = (f, \frac{x^y}{n!}) \therefore (f, x) = 1) (f, x^n) = 1 \Rightarrow f(x) \text{无重因式}.$$

P46, 21

$$g'(x) = \frac{1}{2} [f'(x) + f'(a)] + \frac{x-a}{2} f''(x) - f'(x) \Rightarrow g'(a) = 0$$

又  $g(a) = 0$

$$g''(x) = \frac{1}{2} f'''(x) + \frac{1}{2} f''(x) + \frac{x-a}{2} f'''(x) - f''(x) = \frac{1}{2} (x-a) f'''(x) \Rightarrow g''(a) = 0$$

$$g'''(x) = \frac{1}{2} f'''(x) + \frac{x-a}{2} f^{(4)}(x)$$

$\therefore a$  是  $g(x), g'(x), g''(x), g'''(x)$  根, 且使  $g'''(x)$  的  $k+1$  重根

$\therefore a$  是  $g(x)$  的  $k+3$  重根.

P46, 22

“ $\Leftarrow$ ” 必要性显然 (见定理 6 推论 1)

“ $\Rightarrow$ ” 若  $x_0$  是  $f(x)$  的  $t$  重根,  $t > k$ ,

由定理  $\Rightarrow f^{(k)}(x_0) = 0$

若  $t < k \Rightarrow f^{(k-1)}(x_0) \neq 0$ , 所以矛盾.

P46.23

例如  $f(x) = x^{m+1}$ , 则  $x = 0$  是  $f'(x) = (m+1)x^m$  的  $m$  重根

但  $x = 0$  不是  $f(x)$  的根.

**P46.24** 若  $(x-1)f(x)^n$  则  $(x^n-1) \mid f(x^n)$

证若  $f(x) = (x-1)g(x) + r$  (由上节课命题 2)

$$f(x^n) = (x^n-1)g(x^n) + r = \bar{g}(x) + r \Rightarrow r = 0$$

所以  $x^n - 1 \mid f(x^n)$

P46,25

证明 设  $x^2+x+1$  的两个根  $\varepsilon_1, \varepsilon_2, \varepsilon_i^3 = 1$

$$x^2 + x + 1 = (x - \varepsilon_1)(x - \varepsilon_2)$$

$$\therefore \begin{cases} f_1(\varepsilon_1^3) + \varepsilon_1 f_2(\varepsilon_1^3) = 0 \\ f_2(\varepsilon_2^3) + \varepsilon_2 f_1(\varepsilon_2^3) = 0 \end{cases}$$

$$\text{即} \begin{cases} f_1(1) + \varepsilon_1 f_2(1) = 0 \\ f_1(1) + \varepsilon_2 f_2(1) = 0 \end{cases}$$

$$\Rightarrow f_2(1) = 0 \quad f_1(1) = 0$$

$$\Rightarrow (x-1) \mid f_1(x), f_2(x)$$

P46、26 分解  $x^n - 1$ .

$$\text{设 } \varepsilon_0 = 1 \quad \varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, k = 0, 1, 2, \dots, n-1$$

$$(i) \text{ 在 } \mathbb{C} \text{ 中, } x^n - 1 = \prod_{i=0}^{n-1} (x - \varepsilon_i) = (x-1) \prod_{k=1}^{n-1} (x - (\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}))$$

$$(ii) \text{ 在 } \mathbb{R} \text{ 中, } n \text{ 为奇 } x^n - 1 = (x-1) \prod_{k=1}^{2m-2} (x - \varepsilon_k) = (x-1) \prod_{k=1}^{m-1} (x - \varepsilon_k)(x - \varepsilon_{n-k})$$

$$n = 2m-1 \quad = (x-1) \prod_{k=1}^{m-1} (x^2 - 2 \cos \frac{2k\pi}{n} x + 1)$$

$$\text{当 } n = 2m \text{ 时: } x^n - 1 = (x-1)(x+1) \prod_{k=1}^{m-1} (x^2 - 2 \cos \frac{2k\pi}{n} x + 1)$$

p46,27 求有理根:

$$(1) x^3 - 6x^2 + 15x - 14 = f(x).$$

解: 有理根可能为  $\pm 1$ 、 $\pm 2$ 、 $\pm 7$ 、 $\pm 14$ 。

$\because$  当  $a < 0$  时  $f(a) < 0$ , 所以  $f(x)$  的有理根是可能 1, 2, 7, 14

$$f(1) = -4 \neq 0, f(2) = 0, f(7) = 140 \neq 0, f(14) = 1764 \neq 0, \text{ 只有一个 } x=2$$

$$(2) 4x^4 - 7x^2 - 5x - 1 = f(x).$$

解: 有理根可能为  $\pm 1$ 、 $\pm \frac{1}{2}$ 、 $\pm \frac{1}{4}$ ,  $\because f(1) = -9 \neq 0$ ,  $f(-1) = 1 \neq 0$ ,

$$f(\frac{1}{2}) = -5, f(-\frac{1}{2}) = 0, f(\frac{1}{4}) = -2\frac{43}{64}, f(-\frac{1}{4}) = -\frac{11}{64}$$

所以  $f(x)$  只有一个有理根  $x = -\frac{1}{2}$

$$(3) f(x) = x^5 + x^4 - 6x^3 - 14x^2 - 11x - 3 = f(x).$$

解: 可能有有理根为  $\pm 1$ 、 $\pm 3$ 、 $f(1) = -32$ ,  $f(-1) = 0$ ,  $f(3) = 0$   $f(-3) = -96$

故  $f(x)$  有两个有理根 -1, 3

P46,28

①  $x^2+1$ : 解  $y=y+1, x^2+1=y^2+2y+2$  不可约

②  $x^4-8x^3+12x^2+2$  解取  $p=2$ , 由Eisenstein判别法, 不可约。

③  $x^6+x^3+1$ , 解令  $x=y+1$  则

$x^6+x^3+1=y^6+6y^5+15y^4+21y^3+15y^2+9y+3$  取  $p=3$  即可。

④  $x^p+px+1$  为奇素数

解: 取  $y=x+1, x^3+px+1=y^p+\sum_{i=1}^p (c_p^i y^{p-i} (-1)^i + p(y-1)+1$

$$=y^p+\sum_{i=1}^{p-2} (c_p^i (-1)^i y^{p-i} + 2py - p$$

取  $p$  素数, 即可

⑤  $x^4+4kx+1$   $k$  为整数

解: 令  $x=y+1$ , 则  $f(x)=x^4+4kx+1=y^4+4y^3+6y^2+(4+4k)y+(4k+2)$

取  $p=2$ , 则  $p \nmid 4k+2$ ,

即可由Eisenstein判别法,  $f(x)$  于  $\mathbb{Z}(\mathbb{Q})$  上不可约。

P47.1: 证  $\because f_1, g_1$  都是  $f, g$  的组合, 所以若  $c(x)$  是  $f, g$  的公因式, 则必有  $c(x) \mid f_1, c(x) \mid g_1$ , 为  $f_1, g_1$  的公因式, 即

$$CD\{f(x), g(x)\} \subseteq CD\{f_1(x), g_1(x)\}$$

反过来, 得  $f(x) = \frac{1}{ad-bc}(df_1(x) - bg_1(x)), g(x) = \frac{1}{ad-bc}(-cf_1(x) + ag_1(x))$

$\therefore f, g$  也是  $f_1, g_1$  的组合, 同上理, 有

$$CD\{f_1(x), g_1(x)\} \subseteq CD\{f(x), g(x)\}$$

即,  $f$  与  $g$  和  $f_1$  与  $g_1$  的公因式一致, 最大公因式也一致, 那

$$(f(x), g(x)) = (f_1(x), g_1(x))$$

注: 不可约多项式也称既约定多项式

$f(x) \neq 0, a$ , 则  $f(x)$  不是既约, 则称  $f(x)$  可约

P47.2

证:  $\because d_1(x)f(x) = v_1(x)g(x) \Rightarrow (f(x), g(x)) = d(x)$ .

$$f(x) = f_1(x)d(x), g(x) = g_1(x)d(x)$$

$\therefore u_1(x)f_1(x) + v_1(x)g_1(x) = 1$  带余除法

$$\text{令 } u_1(x) = q_1(x)g_1(x) + u(x) \quad \partial(u) < \partial(g_1(x)) = \partial\left(\frac{g(x)}{g_1(x)} + y(x)\right)$$



$$v_1(x)=q_2(x)f_1(x)+v(x) \quad \partial(v)<\partial(f_1(x))=\partial(f(x)/(f(x), g(x)))$$

$$\text{则 } fu+gv+fg_1q_1+f_1gq_2$$

$$=f(x)u(x)+g(x)v(x)+f_1(x)g_1(x)d(x)(q_1+q_2)=d(x)$$

$$\therefore d-(uf+vg)=f_1g_1d(q_1+q_2), \text{左边次数} < \partial(f_1)+\partial(g_1)+\partial(d) \leq \text{右边次数}$$

$$\text{故左、右两侧只有为 } 0, \quad d-(uf+vg)=0$$

$$u(x)f(x)+v(x)g(x)=d(x)$$

$$\text{且 } \partial(u(x))<\partial(g_1(x)), \partial(v(x))<\partial(f_1(x))$$

P47.3 若  $f(x)$  与  $g(x)$  互素, 则  $\forall m \geq 1, f(x^m)$  与  $g(x^m)$  也互素

证:  $\because f(x)$  与  $g(x)$  互素,  $\therefore \exists u(x), v(x), u(x)f(x)+v(x)g(x)=1$

由推广 令  $\varphi(x)=x^m, u(x^m)f(x^m)+v(x^m)g(x^m)=1$

$\therefore (f(x^m), g(x^m))=1$ , 即  $f(x^m)$  与  $g(x^m)$  互素

P47 补 4

$$\text{由定义有 } (f_1, f_2 \cdots f_s) = ((f_1, \cdots, f_{s-1}), f_s),$$

证明  $\exists d_i(x)$  使得  $u_1f_1+u_2f_2+\cdots+u_sf_s=(f_1, f_2 \cdots f_s)$ .

$$\text{证: 设 } d=(f_1, f_2 \cdots f_s), \quad d_1=(f_1 \cdots f_{s-1}) \quad d'=(d_1, f_s)$$

$$\text{显然 } d|f_s \text{ 及 } d|d_1 \Rightarrow d|d',$$

$$\text{反之, } d' \Rightarrow d|d_1', \quad d'|f_s \Rightarrow d|f_i \quad (\forall i) \Rightarrow d'|d.$$

$$\text{又 } d, d' \text{ 首项系数}=1 \Rightarrow d=d'.$$

证: 由归纳方式  $\exists u_i'$ , 使  $u_1'f_1+\cdots+u_{s-1}'f_{s-1}=d_1$ , 又  $\exists v, u$  使得  $vd_1+u_sf_s=d'$ ,

$$\therefore d=d' = vd_1 + u_sf_s = v \left( \sum_{i=1}^{s-1} u_i' f_i \right) + u_sf_s \quad \text{令 } u_i = vu_i' \quad i=1, 2, \cdots, s-1$$

$$=u_1f_1+\cdots+u_{s-1}f_{s-1}+u_sf_s.$$

P48, 补 5

$$\text{证明 若: } f(x)g(x) \text{ 首项系数都}=1 \quad \text{则 } [f, g] = \frac{fg}{(f, g)}$$

证: 令  $(f, g)=d, f=f_1d, g=g_1d$ , 则  $(f_1, g_1)=1$ , 设  $m(x)=f_1g_1d$

显然①  $f|m, g|m$ , 故  $m$  是一个公倍式

再设②  $f|l, g|l \therefore d|l$ , 令  $l=dl_1, \Rightarrow f_1|l_1, g_1|l_1$

$\because (f_1g_1)=1, \therefore f_1g_1|l_1 \Rightarrow f_1g_1d_1|l$  即  $m|l$

$m$  是  $f, g$  的一个最小公倍式

$$\frac{f(x) \cdot g(x)}{f(x) \cdot g(x)}$$

即证得:  $[f(x), g(x)] = f_1(x)g_1(x)d(x) = (f(x) \cdot g(x))$

p48.7:  $f(x)$  首项系数  $=1, \partial(f(x)) > 0$ , 则  $f(x)$  为某不可约多项式  $p(x)$

的方幂的充要条件是  $\forall g(x)$  或者  $(f, g) = 1$  或者  $\exists m: f(x) | g^m(x)$

证明 " $\Leftarrow$ " 反设不是则  $f(x) = p_1^r(x)h(x)$ , 而  $\partial(h(x)) > 0, p_1(x) + h(x) \Rightarrow$

$(p_1, h) = 1$ , 即  $h \nmid p_1$ , 取  $g(x) = p_1(x)$ , 则  $(f, g) \neq 1$ , 且  $\forall m, f \nmid g^m$ , 否则  $h = p_1^s(x)$ , 矛盾.

" $\Rightarrow$ "  $f = p^r, \forall g(x)$ , 若  $(p, g) = 1 \Rightarrow (p^r, g) = (f, g) = 1$ , 若  $(p, g) \neq 1 \Rightarrow p | g \Rightarrow f | g^r(x)$

p48.8:  $f(x)$  首项系数  $=1, \partial(f(x)) > 0$ , 则  $f(x)$  为某不可约多项式的方幂  $\Leftrightarrow$

$\forall g(x) | h(x)$ , 由  $f | gh \Rightarrow f | g$  或者  $\exists m, f(x) | h^m(x)$

证明 " $\Rightarrow$ " 设  $f = p^r$ , 若  $f | gh, (p, h) = 1 \Rightarrow (p^r, h) = 1 \Rightarrow (f, h) = 1 \Rightarrow f | g$

$$(p, h) \neq 1 \Rightarrow p | h \Rightarrow p^r | h^r \Rightarrow f | h^r(x)$$

" $\Leftarrow$ " 反设不是, 则  $f = p_1^r h$ , 而  $\partial(h) > 0, p_1 + h$ , 令  $g = p_1^r, h = h(x)$ , 则

$f | gh$  却  $f + g, f + h^m, \forall m (\because (p, h) = 1 \Rightarrow (p^r, h^m) = 1)$

P48, 补 9 证:  $x^n + ax^{n-m} + b$  没有重数  $> 2$  的非零根

证: 反设  $f(x) = x^n + ax^{n-m} + b$  有  $k$  重根  $\alpha$ , ( $k > 2, \alpha \neq 0$ )

$g(x) = f'(x) = nx^{n-1} + a(n-m)x^{n-m-1}$  有  $k$  重根  $\alpha \neq 0$

$$\Rightarrow nx^{n-m-1}(x^m + \frac{a(n-m)}{n}) \text{ 有重根 } \alpha \neq 0$$

$\therefore h(x) = x^m + \frac{a(n-m)}{n}$  有重根  $\alpha \neq 0$

$$\text{但 } h'(x) = mx^{m-1} \quad \therefore (h(x), h'(x)) = \begin{cases} 1 \\ h'(x) \end{cases} \begin{cases} \text{无重数根} \\ \text{重根 } 0 \end{cases}$$

P48, 补 10

$0 \neq f(x) \in C[x]$ , 且  $f(x) | f(x^n), n > 1$ ,

证明  $f(x)$  的根只能为 0 或单位根 (即满足某  $x^m = 1$  的根).

证: 设  $\alpha$  为  $f(x)$  的根, 由  $f(x^n) = f(x)g(x)$

$\therefore f(\alpha^n) = 0, \alpha^n$  为  $f(x)$  的根,

$\therefore f(\alpha^{n^2}) = 0, \alpha^{n^2}$  为  $f(x)$  的根,

$\Rightarrow \alpha, \alpha^n, \alpha^{n^2}, \alpha^{n^3}, \dots$  都为  $f(x)$  的根.

$\because f(x) \neq 0, \therefore f(x)$  不可能有无限个根, 其中必有相等者:

$$\alpha^{n^i} = \alpha^{n^j} \quad (\text{不妨设 } i > j),$$

$$\therefore \alpha^{n^2} (\alpha^{n^i - n^j} - 1) = 0, \text{ 令 } n^i - n^j = m.$$

则或  $\alpha = 0$ , 或  $\alpha$  是  $x^m = 1$  的根.

P48、补 11 题:

$$\because f'(x) \mid f(x) \Rightarrow f(x) = a(x-b)^n \therefore f(x) \text{ 有 } n \text{ 重根 } b.$$

补充 P48 12 题:  $a_1, a_2, \dots, a_n$  的两两不同.  $F(x) = (x-a_1)(x-a_2)\dots(x-a_n)$

证: (1)

$$\sum_{i=1}^n \frac{F(x)}{(f-a_i)F'(a_i)} = 1 \quad \because l_i = \frac{F(x)}{(x-a_i)F'(a_i)} = \frac{(x-a_1)(x-a_{i-1})(x-a_{i+1})\dots(x-a_n)}{(a_i-a_1)(a_i-a_{i-1})(a_i-a_{i+1})\dots(a_i-a_n)}$$

$$l_i(a_j) = 0, l_i(a_i) = 1, \therefore \sum_{i=1}^n l_j(a_j) = 1, \forall j = 1 \dots n. \forall i, \sum_{i=1}^n l_i(x_i), i = 1 \dots n$$

为  $n-1$  次多项式,  $\therefore \sum_{i=1}^n l_i(x) = 1$ , (2) 设  $f(x) = F(x)q(x) + r(x)$ , 则  $f(a_i) = r(a_i)$ ,

$$\text{而 } n-1 \text{ 形式多项式 } \sum_{r=1}^n f(a_i)l_i(x) = h(x) : h(a_j) = f(a_j)$$

$$= r(a_j) \quad \therefore h(x) = r(x)$$

p49、补 13 题:

$$(1) \text{ 求 } f(x) \quad \partial(f(x)) < 4 \text{ 且 } f(2) = 3 \quad f(3) = -1 \quad f(4) = 0 \quad f(5) = 2$$

$$l_1(x) = \frac{(x-3)(x-4)(x-5)}{(2-3)(2-4)(2-5)} = -\frac{1}{6}(x^3 - 12x^2 + 47x - 60)$$

$$l_2(x) = \frac{(x-2)(x-4)(x-5)}{(3-2)(3-4)(3-5)} = \frac{1}{2}x^3 - \frac{11}{2}x^2 + 19x - 20$$

$$l_3(x) = \frac{(x-2)(x-3)(x-5)}{(4-2)(4-3)(4-5)} = \frac{1}{2}x^3 + 5x^2 + \frac{31}{2}x + 15$$

$$l_4(x) = \frac{(x-2)(x-3)(x-4)}{(5-2)(5-3)(5-4)} = \frac{1}{6}x^3 + \frac{3}{2}x^2 + \frac{13}{3}x - 4$$

$$\therefore f(x) = \sum_{i=1}^4 (l_i(x))f(a_i) = 3l_1 - l_2 + 0 \cdot l_3 + 2l_4 = -\frac{2}{3}x^3 + \frac{17}{2}x^2 - \frac{203}{6}x + 42$$

②求一个二次多项式 $f(x)$ ,  $f(0) = \sin 0, f(\frac{\pi}{2}) = \sin \frac{\pi}{2}, f(\pi) = \sin \pi = 0$ .

$$l_1(x) = \cdots,$$

$$l_2 = \frac{(x-0)(x-\pi)}{(\frac{\pi}{2}-0)(\frac{\pi}{2}-\pi)},$$

$$l_3 = \cdots,$$

$$\therefore f(x) = f(\frac{n}{2})l_2 = \frac{x(x-\pi)}{-\frac{\pi^2}{4}}$$

③ $f(x)$ 可能低次项: $f(0)=1 \quad f(1)=2 \quad f(2)=5 \quad f(3)=10$

$$l_1(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = -\frac{1}{6}x^3 + x^2 - \frac{11}{6}x + 1$$

$$l_2(x) = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{1}{2}x^3 - \frac{5}{2}x + 3x$$

$$l_3(x) = \frac{(x-0)(x-2)(x-3)}{(2-0)(2-1)(2-3)} = -\frac{x^3}{2} + 2x^2 + 3x$$

$$l_4(x) = \frac{(x-0)(x-2)(x-3)}{(3-0)(3-1)(3-2)} = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x$$

$$\therefore f(x) = l_1(x) + 2l_2(x) + 5l_3(x) + 10l_4(x) = x^2 + 1$$

P49.补 14),  $f(x) \in \mathbb{Z}[x], f(0), f(1)$  奇, 则 $f(x)$  无整数根.

证: 反设 $f(x)$  有整数数根 $m$ , 则 $x-m \mid f(x)$ ,

$f(0)$  奇  $\Rightarrow -m$  奇  $f(1)$  奇  $\Rightarrow 1-m$  奇 矛盾!

## 第二章 行列式习题解答

P96.1 ① $\tau(134782695) = 0+1+1+3+3+0+1+1 = 10 \therefore 13478695$  偶排列

② $\tau(217986354) = 1+0+4+5+4+3+0+1 = 18 \therefore 21798354$  偶排列

③ $\tau(98765432) = 8+7+6+\cdots+1+2+1 = 36 \therefore 987654321$  偶排列

P96.2 ①若 1274i56k9 偶则  $i, k=3.8$  或  $8.3$

$$\tau(127435689)=5, \tau(127485639)=10 \therefore i=8, k=3$$

②若 1i25j4897 奇则 i, k=3, 6 或 6.3

$$\tau(132564897)=4 \quad \tau(162534897)=7 \quad \therefore i=6 \quad k=3$$

P96.3  $3 \ 1 \ 2 \ 4 \ 3 \ 5 \rightarrow 2 \ 1 \ 4 \ 3 \ 5 \rightarrow 2 \ 5 \ 4 \ 3 \ 1 \rightarrow 2 \ 5 \ 3 \ 4 \ 1$  即得

$$P96.4 \quad \because \tau(n(n-1), \dots, 3 \ 2 \ 1) = C_n^2 = \frac{n(n-1)}{2}$$

$\therefore$  当  $4|n$  或  $4|n-1$  即  $n=4k$  或  $n=4k+1$  时,  $C_n^2$  为偶数, 偶排列。

当  $n=4n+2$ ,  $n=4n+3$ , 则  $C_n^2$  为奇数, 是奇排列

P96.5 排列  $\pi_1: X_1 X_2 \dots X_n$  与  $\pi_2: X_n X_{n-1} \dots X_2 X_1$  中, 任取两个数  $X_i, X_j$

若  $X_i, X_j$  在  $\pi_1$  中有逆序, 则在  $\pi_2$  中没有, 反之在  $\pi_1$  中没有逆序, 则  $\pi_2$  中有逆序,  $\therefore \tau(\pi_1) +$

$$\tau(\pi_2) = C_n^2$$

$$\text{即 } \tau(X_n X_{n-1} \dots X_2 X_1) = C_n^2 - \tau(X_1 X_2 \dots X_n).$$

P97.6. 由于  $\tau(234516) + \tau(312645) = 8 \cdot a_{23} a_{31} a_{42} + a_{56} a_{14} a_{65}$  带正号

$$\text{由 } \tau(341562) + \tau(234165) = 10 \therefore a_{32} a_{43} a_{14} a_{51} a_{66} a_{23} \text{ 带正号}$$

P96.7  $j_1 j_2 j_3 j_4$  由于  $j_2=3$ ,  $\therefore j_1 j_3 j_4$  取 1、2、4 的排列

$$j_1 j_3 j_4 = 124$$

$$\Rightarrow \tau(1324) = 1, j_1 j_3 j_4 \Rightarrow \tau(1342) = 2; j_1 j_3 j_4 = 214 \Rightarrow \tau(2314) = 2$$

$$j_1 j_3 j_4 = 241 \Rightarrow \tau(2341) = 3; j_1 j_3 j_4 = 142 \Rightarrow \tau(1342) = 2; j_1 j_3 j_4 = 214 \Rightarrow \tau(2314) = 2$$

$$j_1 j_3 j_4 = 241 \Rightarrow \tau(2341) = 3; j_1 j_3 j_4 = 412 \Rightarrow \tau(4312) = 5; j_1 j_3 j_4 = 421 \Rightarrow \tau(4321) = 6$$

$$\therefore \text{取负号只有 } -a_{11} a_{23} a_{32} a_{44}, -a_{12} a_{23} a_{34} a_{41}, -a_{14} a_{23} a_{31} a_{42}$$

$$P97.8(1) D = (-1)^{\tau(n \dots 321)} 123 \dots (n-1)n = (-1)^{\frac{n(n-1)}{2}} \cdot n$$

$$P97.8(2) D = (-1)^{\tau(23 \dots n1)} 123 \dots n = (-1)^{n-1} \cdot n1$$

$$P97.8(3) D = (-1)^{\tau((n-1) \dots (n-2) \dots 21n)} 123 \dots n = (-1)^{\frac{(n-1)(n-2)}{2}} \cdot n$$

$$P97.9 D = \sum_{j_1 j_2 j_3 j_4 j_5} (-1)^{(j_1 j_2 j_3 j_4 j_5)} a_{j_1} b_{j_2} c_{j_3} d_{j_4} e_{j_5}$$

因后三行后三列为 0, 所以非零的项只有,  $j_3 \leq 2, j_4 \leq 2, j_5 \leq 2$ , 而  $j_3, j_4, j_5$  是互不相

同的数, 这是不可能的, 所以没有非 0 的项,  $D=0$

$$\text{P97.10} \quad f(x) = \begin{vmatrix} 2x & x & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} \text{求 } x^4, x^3 \text{ 的系数}$$

解:  $\because$  行列式中每项由每行出一元相乘, 故  $x^4$  必须将 2. 3. 4 行的  $x$  都取, 这时第  $i$  行取第  $i$  列, 这是行列式的一项,  $ax^4$ , 系数为  $a=2$ 。

$x^3$  项必有一元  $a_{ij}$  在对角线外, 于是  $i$  行,  $j$  列的  $x$  不能再取了, 故当  $i=1, j>2$  时, 至少去掉 3 个  $x$ , 不含  $x^3$  项了, 对于  $i>2, j=2$  同理

其它情形, 至少去掉两个  $x$  且第一行 (或第二列) 的两个  $x$  只能取一个, 故不含  $x^3$  项, 只剩下  $i=1, j=2$  时,  $a_{12}$  本身是  $x$  项为

$$(-1)^{(2134)} a_{12} a_{21} a_{33} a_{44} = -x 1 x x = -x^3, \text{ 系数为 } -1$$

$$\text{P97.11} \quad d = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1, j_2, \cdots, j_n)} \cdot 1 = 0 \quad \text{故 } \sum \text{ 中 } +1 \text{ 与 } -1 \text{ 一样多, 即 } + \text{ 号, } - \text{ 号一样多,}$$

也即奇偶排列一样多,  $\therefore n \geq 2$  时, 奇偶排列各占一半.

$$\text{P98.12} \textcircled{1} \quad p(x) = \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \end{vmatrix} = V x^{n-1} + \cdots \text{(按第一行展开)}$$

$$\because a_1 a_2, \cdots a_{n-1} \text{ 两两不同 } \therefore V_{n-1} \neq 0 \text{ 即 } \partial(p(x)) = n-1$$

$$\because p(a_1) = p(a_2) = \cdots = p(a_{n-1}) = 0 \text{ (总有两行相同) 最多 } n-1 \text{ 个根,}$$

$$\textcircled{2} \text{ 即 } p(x) \text{ 的所有根为 } a_1, a_2, \cdots, a_{n-1}$$

$$\text{P98.13} \textcircled{2} \quad \xleftrightarrow[2 \times (1)+3]{1 \times (1)+3} \begin{vmatrix} x & x & x+y \\ y & x+y & x \\ 2(x+y) & 2(x+y) & 2(x+y) \end{vmatrix} \xleftrightarrow[2(x+y)]{3(2(x+y))^{-1}} 2(x+y)$$

$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ 1 & 1 & 1 \end{vmatrix} \xleftrightarrow[3(-x)+2]{3(-x-y)+1} 2(x+y) \begin{vmatrix} -y & -x & 0 \\ y-x & y & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x+y)[-y^2 + xy - x^2] = -2(x^3 + y^3)$$

$$P48.13③ \begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} \xrightarrow{P2.23.3.4(例)} (3 + (4-1) \cdot 1)(3-1)^{4-1} = 6 \cdot 2^3 = 48$$

P98.13④ (法一):

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & -1 & -8 & -10 \\ 0 & -7 & -10 & -13 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 4 & 36 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 40 \end{vmatrix} = 160$$

法二:

$$= \begin{vmatrix} 10 & 2 & 3 & 4 \\ 10 & 3 & 4 & 7 \\ 10 & 4 & 1 & 2 \\ 10 & 1 & 2 & 3 \end{vmatrix} = 10 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & -2 & -2 \\ 0 & -1 & -1 & -1 \end{vmatrix} = 20 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{vmatrix} = 20 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -1 & -1 \end{vmatrix}$$

$$= (-20) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -4 \end{vmatrix} = 160$$

法三:

$$f(x) = 1 + 2x + 3x^2 + 4x^3 \quad \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \text{ 为 4 次单位根 } \pm 1, \pm i$$

$$\text{令 } \varepsilon_1 = 1, \varepsilon_2 = -1, \varepsilon_3 = i, \varepsilon_4 = -i, \text{ 则}$$

$$\text{行列式 } d = (-1)^{c_{4-1}} f(1)f(-1)f(i)f(-i)$$

$$= (-1)^3 \cdot 10 \cdot (-2) \cdot (-2 - 2i) \cdot (-2 + 2i) = 20[(-2)^2 - (2i)^2] = 160$$

令

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1+x & 1 & 1 \\ 0 & 1 & 1-x & 1 \\ 0 & 1 & 1 & 1+y \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & x & 0 & 0 \\ -1 & 0 & -x & 0 \\ -1 & 0 & 0 & y \end{vmatrix} = x^2 y^2$$

P98.13⑤解:

$$\text{解法二, 设 } f(x, y) = \begin{vmatrix} 1+x & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} \quad \text{则第一行减第二行} \quad \begin{vmatrix} x & x & 0 & 0 \\ & & * & \\ & & & \\ & & & \end{vmatrix}$$

$$\therefore x \mid f(x, y)$$

$$\text{又因为 } f(-x, y) = \begin{vmatrix} 1-x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} \xrightarrow[\text{再交换1,2列}]{\text{交换1,2行}} \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = f(x, y).$$

$\therefore f(x, y)$  关于  $x$  是偶函数, 即  $x^2 \mid f(x, y)$ .

同理,  $y \mid f(x, y)$ , 且也是偶函数, 所以  $y^2 \mid f(x, y)$

$$\therefore \partial(f(x, y)) \leq 4 \quad \therefore x^2 y^2 \mid f(x, y) \Rightarrow f(x, y) = kx^2 y^2$$

而  $f(x, y)$  中  $x^2 y^2$  的系数为 1. 故有  $f(x, y) = x^2 y^2$ .

P98 13⑥

$$\xrightarrow[\begin{smallmatrix} 1 \times (-1) + 2 \\ 1 \times (-1) + 3 \\ 1 \times (-1) + 4 \end{smallmatrix}]{\quad} \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix} \xrightarrow[\begin{smallmatrix} 2 \times (-2) + 3 \\ 2 \times (-3) + 4 \end{smallmatrix}]{\quad} \begin{vmatrix} a^2 & 2a+1 & 2 & 6 \\ b^2 & 2b+1 & 2 & 6 \\ c^2 & 2c+1 & 2 & 6 \\ d^2 & 2d+1 & 2 & 6 \end{vmatrix} \xrightarrow[\text{性质5}]{\quad} 0$$

$$\xrightarrow[\begin{smallmatrix} 2 \times (-1) + 3 \end{smallmatrix}]{\quad} \begin{vmatrix} 246 & 427 & -100 \\ 1014 & 543 & -100 \\ -342 & 721 & -100 \end{vmatrix} \xrightarrow[\begin{smallmatrix} 3 \times (246) + 1 \\ 3 \times (427) + 2 \end{smallmatrix}]{\quad} \begin{vmatrix} 0 & 0 & 1 \\ 768 & 116 & 1 \\ -588 & 294 & 1 \end{vmatrix} \xrightarrow[\begin{smallmatrix} 2 \times (2) + 1 \end{smallmatrix}]{\quad} \begin{vmatrix} 0 & 0 & 1 \\ 1000 & 116 & 1 \\ 0 & 294 & 1 \end{vmatrix} \times (-100) \xrightarrow[\text{性质7}]{\quad} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 116 & 1000 \\ 1 & 294 & 0 \end{vmatrix} \times (100) = -100 \times \begin{vmatrix} 1 & 0 & 0 \\ 1 & 294 & 0 \\ 1 & 116 & 1000 \end{vmatrix} = -29400000$$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1000 & 116 & 1 \\ 0 & 294 & 1 \end{vmatrix} \times (-100) \xrightarrow[\text{性质7}]{\quad} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 116 & 1000 \\ 1 & 294 & 0 \end{vmatrix} \times (100) = -100 \times \begin{vmatrix} 1 & 0 & 0 \\ 1 & 294 & 0 \\ 1 & 116 & 1000 \end{vmatrix} = -29400000$$

$$P98.14 \quad \text{左} \xrightarrow[\begin{smallmatrix} 2 \times (1) + 1 \\ 3 \times (1) + 1 \\ 1 \times (\frac{1}{2}) \end{smallmatrix}]{\quad} 2 \begin{vmatrix} a+b+c & c+a & a+b \\ a_1+b_1+c_1 & c_1+a_1 & a_1+b_1 \\ a_2+b_2+c_2 & c_2+a_2 & a_2+b_2 \end{vmatrix} \xrightarrow[\begin{smallmatrix} 1 \times (-1) + 2 \\ 1 \times (-1) + 3 \end{smallmatrix}]{\quad} 2 \begin{vmatrix} a+b+c & -b & -c \\ a_1+b_1+c_1 & -b_1 & -c_1 \\ a_2+b_2+c_2 & -b_2 & -c_2 \end{vmatrix}$$

$$\xrightarrow[\begin{smallmatrix} 2 \times (1) + 1 \\ 2 \times (1) + 1 \end{smallmatrix}]{\quad} 2 \begin{vmatrix} a & -b & -c \\ a_1 & -b_1 & -c_1 \\ a_2 & -b_2 & -c_2 \end{vmatrix} \xrightarrow[\begin{smallmatrix} 2 \times (-1) \\ 3 \times (-1) \end{smallmatrix}]{\quad} \text{右} (-1)^2 = \text{右}$$

P98. 15 求出所有代数余子式



$$\textcircled{1} \begin{vmatrix} 1 & 2 & 1 & 4 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{vmatrix} \text{直接计算有} \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} = \begin{pmatrix} -6 & 0 & 0 & 0 \\ -12 & 6 & 0 & 0 \\ 15 & -6 & -3 & 0 \\ 7 & 0 & 1 & -2 \end{pmatrix}$$

$$\textcircled{2} \begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & 1 \\ 0 & 1 & 4 \end{vmatrix} \text{直接计算有} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 7 & -12 & 3 \\ 6 & 4 & -1 \\ -5 & 5 & 5 \end{pmatrix}$$

P99.16①

$$= - \begin{vmatrix} -1 & +3 & +5 & 1 & 2 \\ 2 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & -1 & 4 \\ 3 & 3 & 1 & 2 & 1 \\ 2 & 1 & 0 & 3 & 5 \end{vmatrix} \begin{matrix} 1 \times (2) + 2 \\ 1 \times (3) = 4 \\ 1 \times (2) + 5 \end{matrix} \begin{vmatrix} -1 & 3 & 5 & 1 & 2 \\ 0 & 6 & 11 & 4 & 5 \\ 0 & 1 & 2 & -1 & 4 \\ 0 & 12 & 16 & 5 & 7 \\ 0 & 7 & 10 & 5 & 9 \end{vmatrix} \begin{matrix} 2(-)3 \\ 2(-)1 \end{matrix} \begin{vmatrix} -1 & 3 & 5 & 1 & 2 \\ 0 & -1 & -2 & 1 & 4 \\ 0 & 6 & 11 & 4 & 5 \\ 0 & 12 & 16 & 5 & 7 \\ 0 & 7 & 10 & 5 & 9 \end{vmatrix}$$

$$\begin{matrix} 2 \times (6) + 3 \\ 2 \times (2) + 4 \\ 2 \times (7) + 5 \end{matrix} \begin{vmatrix} -1 & 3 & 5 & 1 & 2 \\ 0 & -1 & -2 & 1 & -4 \\ 0 & 0 & -1 & -1 & 9 \\ 0 & 0 & -8 & 17 & -41 \\ 0 & 0 & -4 & 12 & -19 \end{vmatrix} \begin{matrix} 3(-)1 \\ 3 \times (8) + 4 \\ 3 \times (4) + 5 \end{matrix} \begin{vmatrix} -1 & 3 & 5 & 1 & 2 \\ 0 & -1 & -2 & 1 & -4 \\ 0 & 0 & -1 & -1 & 9 \\ 0 & 0 & -8 & 63 & 111 \\ 0 & 0 & 0 & -28 & 57 \end{vmatrix} \begin{matrix} 4 \times (-2) + 3 \end{matrix}$$

$$\begin{vmatrix} -1 & 3 & 5 & 1 & 2 \\ 0 & -1 & -2 & 1 & -4 \\ 0 & 0 & -1 & -10 & 19 \\ 0 & 0 & 0 & -7 & -3 \\ 0 & 0 & 0 & -28 & 57 \end{vmatrix}$$

$$\begin{matrix} 1 \times (-1) \\ 2 \times (-1) \\ 4 \times (-4) + 5 \end{matrix} \begin{vmatrix} 1 & - & -5 & -1 & -2 \\ 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -10 & 19 \\ 0 & 0 & 0 & -10 & -3 \\ 0 & 0 & 0 & 0 & 69 \end{vmatrix} = (-7)69 = -483$$

P99.16②

$$\begin{array}{l} 1 \times (2) \\ 5 \times (2) \frac{1}{8} \\ 4 \times (2) \end{array} \begin{vmatrix} 2 & 1 & 0 & 4 & -2 \\ 2 & 0 & -1 & 2 & 2 \\ 3 & 2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 & 2 \\ 4 & 2 & 6 & 0 & 1 \end{vmatrix} \begin{array}{l} 1 \rightarrow 4 \\ 1 \times (2) + 2 \\ 1 \times (-3) + 3 \\ 1 \times (2) + 4 \\ 1 \times (-4) + 5 \end{array} \begin{vmatrix} 1 & -1 & 0 & 2 & 2 \\ 0 & 2 & -1 & -2 & -2 \\ 0 & 5 & 1 & -5 & -6 \\ 0 & 3 & 0 & 0 & -6 \\ 0 & 6 & 6 & -8 & -7 \end{vmatrix}$$

$$\begin{array}{l} 2-4 \\ 4 \times (-1) + 2 \end{array} + \frac{1}{8} \begin{vmatrix} 1 & -1 & 0 & 2 & 2 \\ 0 & 2 & 1 & 2 & -4 \\ 0 & 5 & 1 & -5 & -6 \\ 0 & 2 & -1 & -2 & -2 \\ 0 & 6 & 6 & -8 & -7 \end{vmatrix}$$

$$\begin{array}{l} 2 \times (-5) + 3 \\ 2 \times (-2) + 4 \\ 2 \times (-8) + 5 \end{array} \begin{vmatrix} 1 & -1 & 0 & 2 & 2 \\ 0 & 2 & 1 & 2 & -4 \\ 0 & 0 & -4 & -15 & -14 \\ 0 & 0 & -3 & -6 & 6 \\ 0 & 0 & 0 & -20 & 17 \end{vmatrix} \begin{array}{l} 4-3 \\ 2 \times \frac{1}{3} \\ 3 \times (-4) + 4 \end{array} - \frac{3}{8} \begin{vmatrix} 1 & -1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & -7 & 6 \\ 0 & 0 & 0 & 0 & -\frac{1}{7} \end{vmatrix}$$

$$= -\frac{3}{8} \cdot 1 \cdot 1 \cdot (-1) \cdot 7 \cdot \left(\frac{1}{7}\right) = \frac{3}{8}$$

P99.17①若

$j_1 j_2 \cdots j_n$  中  $j_n = n$ , 则  $j_{n-1}$  取  $n-1$ ,  $j_{n-2}$  取  $n-2 \cdots, j_2 = 2, j_1 = 1$  或若  $j_n = 1$ , 则  $\Rightarrow j_1 = 2, j_2 = 3, \cdots j_{n-1} = n$ . 故只有两项.  $\tau(123 \cdots n) = 0, \tau(2, 3, \cdots n1) = n-1$

$$\therefore d = \sum x^n + (-1)^{n-1} y^n$$

P99.17②

$n=1$  则  $d = a_1 - b_1$

$n=2$ , 则  $d = (a_1 - b_1)(a_2 - b_2) - (a_1 - b_2)(a_2 - b_1)$

$$= a_1 b_1 - a_2 b_2 + a_1 b_1 - b_2 a_2$$

$$= (a_2 - a_1)(b_2 - b_1)$$

$$\text{当 } n \geq 3 \text{ 时 } \begin{vmatrix} a_1 - b_1 & b_1 - b_2 & \vdots & a_1 - b_1 \\ a_2 - b_2 & b_1 - b_2 & \vdots & b_1 - b_n \\ \vdots & \vdots & \vdots & b_1 - b_n \\ a_n - b_1 & b_1 - b_n & \vdots & b_1 - b_n \end{vmatrix} = 0 \text{ 第2列与第 } n \text{ 列成比例}$$

$$P99.17③ \xleftrightarrow[\substack{2 \times (1)+1 \\ 3 \times (1)+1 \\ n \times (1)+1}]{\quad} \begin{vmatrix} \sum_{i=1}^m x_i - m & x_2 & \vdots & x_n \\ \vdots & x_2 - m & \vdots & \vdots \\ \vdots & \vdots & \vdots & x_n \\ \sum_{i=1}^m x_i - m & x_2 & \vdots & x_{n-m} \end{vmatrix} \xleftrightarrow[\substack{1(\sum_{i=1}^n (x_{i-m})^{-1}) \\ 1 \times (-x_2)+2 \\ 1 \times (-x_3)+3 \\ 1 \times (-x_n)+n}]{\quad} \begin{vmatrix} 1 & 0 & \vdots & 0 \\ 1 & -m & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ 1 & 0 & \vdots & -m \end{vmatrix}$$

$$(\sum_{i=1}^h X_i - m) = (\sum_{i=1}^h X_i - m)(-m)^{n-1} = (-1)^n (m - \sum_{i=1}^h X_i) m^{n-1}.$$

$$P99.17 \text{ ④} \xleftrightarrow[\substack{2 \times (-1)+3 \\ 2 \times (i-4)+4}]{\quad} \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & \cdots & \cdots & n-2 \end{vmatrix} \xleftrightarrow[\substack{\text{性质} \\ 1 \leftrightarrow 2}]{\quad} \begin{vmatrix} 2 & 2 & \cdots & 2 \\ -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ & & & 2 \\ & & & & n-2 \end{vmatrix}$$

$$\xleftrightarrow[\substack{1(\frac{1}{2}) \\ 1 \leftrightarrow 2}]{\quad} 2 \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & \\ & 0 & 1 & & 0 \\ & & 0 & & \\ & 0 & 0 & & n-2 \end{vmatrix}$$

$$= -2 \cdot (n-2)! \quad (n \geq 2), \text{ 且 } n=1 \text{ 时, } D=1 \text{ (左上角 } 1)$$

P99.17.5从最后一列开始, 第n列加到第n-1列, 再第n-1列加到第n-2列..., 第2列加

$$= \begin{vmatrix} \frac{n(n+1)}{2} & \frac{n(n+1)}{2} & \frac{n(n-1)}{2} - 3 & \cdots & 2n-1-n \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1-n \end{vmatrix}$$

$$= \frac{n(n+1)}{2} \cdot (-1)(-2) \cdots (1-n) = (-1)^{n-1} \cdot \frac{1}{2} \cdot (n+1)!$$

P100.18①:从第二列起: 有列 (第三列)

乘以  $a_{i-1}$  加到第一列, 则有

$$\begin{aligned}
D &= \begin{vmatrix} a_o \sum_{i=2}^{n+1} \frac{1}{a_{i-1}} & 1 & 1 & \cdots & 1 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a_n \end{vmatrix} \\
&= a_1 a_2 \cdots a_n \left( a_o - \sum_{i=2}^{n+1} \frac{1}{a_{i-1}} \right) = a_1 a_2 \cdots a_n \left( a_o - \sum_{i=1}^n \frac{1}{a_i} \right) \\
&= a_o a_1 \cdots a_n + \sum_{i=1}^n a_1 \cdots a_{i-1} a_{i+1} \cdots a_n, \quad (\alpha_i \neq 0) \quad ②
\end{aligned}$$

P100.18④

$$D_n = \begin{vmatrix} \cos a & 1 & & & \\ 1 & 2 \cos a & 1 & & \\ & 1 & 2 \cos a & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2 \cos a \end{vmatrix} = \cos n\alpha$$

证法一，用归纳法， $D_1$ 成立，

设 $k < n$ 时 $D_k = \cos k\alpha$ , 当 $k = n$ 时, 因为

$$\begin{aligned}
D_n &= \cos 2\alpha D_{n-1} - 1 \cdot D_{n-2} = (2 \cos \alpha \cdot \cos(n-1)\alpha - \cos(n-2)\alpha) = (\cos n\alpha + \cos(n-2\alpha - \cos(n-2)\alpha)) \\
&= \cos n\alpha. \text{证毕}
\end{aligned}$$

证法二：

$$\because D_n = 2 \cos \alpha D_{n-1} - D_{n-2} \quad \text{找不出适当倍数左移.} (i^2 = -1),$$

$$D_n - (\cos \alpha + i \sin \alpha) D_{n-1} = (\cos \alpha - i \sin \alpha) [D_{n-1} - (\cos \alpha + i \sin \alpha) D_{n-2}]$$

$$\text{同理 } D_n = (\cos \alpha - i \sin \alpha) D_{n-1} = (\cos(n-1)\alpha + i \sin(n-1)\alpha)(i \sin \alpha)$$

$$\text{相减: } 2i \sin \alpha D_n = i \sin \alpha [\cos n\alpha + \sin n\alpha + \cos n\alpha - \sin n\alpha]$$

$$\text{即 } D_n = \frac{1}{2} (2 \cos n\alpha) = \cos n\alpha$$

P100.18⑤以第一行 $\times (-1)$ 加到后面各行

$$= \begin{vmatrix} 1+a_1 & 1 & 1 & \cdots & 1 \\ -a_1 & a_2 & 0 & \cdots & 0 \\ -a_1 & 0 & a_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & \cdots & 0 & a_n \end{vmatrix}$$

$$(\text{属于交行列式}) = a_1 \cdots a_n (1 + a_1 - \sum_{i=2}^n \frac{-a_1}{a_i})$$

$$= a_1 a_2 \cdots a_n (\frac{1}{a_1} + 1 + \sum_{i=2}^n \frac{1}{a_i})$$

$$\text{要求 } (a_i \neq 0) \quad = a_1 a_2 \cdots a_n (1 + \sum_{i=2}^n \frac{1}{a_i})$$

p101.19①

$$D = \begin{vmatrix} 2 & -2 & 3 & 2 \\ 3 & -3 & 3 & 2 \\ 3 & -3 & -1 & 2 \\ 3 & -1 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 & 2 \\ -3 & 0 & -6 & -4 \\ 1 & 0 & -4 & 0 \\ 1 & 0 & 0 & -3 \end{vmatrix} = \begin{vmatrix} -3 & -6 & -4 & 0 \\ 0 & -4 & 0 & 0 \\ 1 & 0 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -6 & -13 \\ 0 & -4 & 3 \\ 1 & 0 & -3 \end{vmatrix} = -18 - 52 = -70$$

$$\text{且 } D_1 = D_2 = D_3 = D_4 = -70 \quad \therefore \quad x_i = \frac{D_i}{D} = 1 \quad i = 1, 2, 3, 4$$

P101, 19②

$$D = \begin{vmatrix} 1 & 2 & 3 & -2 \\ 2 & -1 & -2 & -3 \\ 3 & 2 & -1 & 2 \\ 2 & -3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & -2 \\ 0 & -5 & -8 & 1 \\ 0 & -4 & -10 & 8 \\ 0 & -7 & -4 & 5 \end{vmatrix} = \begin{vmatrix} 5 & 8 & 1 \\ 4 & 10 & 8 \\ 7 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -36 & -54 & 8 \\ -18 & -36 & 5 \end{vmatrix} = \begin{vmatrix} 36 & 54 \\ 18 & 36 \end{vmatrix}$$

$$= 18^2 \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 18^2 = 324$$

$$\text{且 } D_1=324, \quad D_2=648, \quad D_3=-324, \quad D_4=-648,$$

$$\therefore x_1 = \frac{D_1}{D} = 1, x_2 = \frac{D_2}{D} = 2, x_3 = -1, x_4 = -2$$

$$\text{P101 19 (3), 即 } D = \begin{vmatrix} 1 & 2 & -2 & 4 & -1 \\ 2 & -1 & 3 & -4 & 2 \\ 3 & 1 & -1 & 2 & -1 \\ 4 & 3 & 4 & 2 & 2 \\ 1 & -1 & -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 3 & -1 & 2 & 2 \\ 0 & 1 & 5 & -8 & 8 \\ 0 & 4 & 2 & -4 & 8 \\ 0 & 7 & 8 & -6 & -14 \\ 1 & -1 & -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 0 & -16 & 26 & -22 \\ 1 & 5 & -8 & 8 \\ 0 & -18 & 38 & -24 \\ 0 & -27 & 50 & -42 \end{vmatrix}$$

$$= - \begin{vmatrix} 2 & -2 & 2 \\ -18 & 28 & -24 \\ -9 & 22 & -18 \end{vmatrix} = - \begin{vmatrix} 0 & -2 & 0 \\ 10 & 28 & 4 \\ 13 & 22 & 4 \end{vmatrix} = -(-2) \cdot (x-1)^{1+2} \begin{vmatrix} 10 & 4 \\ 13 & 4 \end{vmatrix} = -8 \begin{vmatrix} 10 & 1 \\ 13 & 1 \end{vmatrix} = 24$$

同理算出  $d_1 = 96, d_2 = -336, d_3 = -96, d_4 = 169, d_5 = 312$

即得  $x_1 = -4, x_2 = -14, x_3 = -4, x_4 = 7, x_5 = 13$

(消元法解)

$$\bar{A} = \begin{pmatrix} 1 & 2 & -2 & 4 & -1 & -1 \\ 2 & -1 & 3 & -4 & 2 & 8 \\ 3 & 1 & -1 & 2 & -1 & 3 \\ 4 & 3 & 4 & 2 & 2 & -2 \\ 1 & -1 & -1 & 2 & -3 & -3 \end{pmatrix} \xrightarrow{\substack{\text{5行移最上} \\ \text{再相减}}} \begin{pmatrix} 1 & -1 & -1 & 2 & -3 & -3 \\ 0 & 1 & -1 & 2 & 2 & 2 \\ 0 & 1 & 5 & -8 & 8 & 14 \\ 0 & 4 & 2 & -4 & 8 & 12 \\ 1 & 7 & 8 & -6 & 14 & 10 \end{pmatrix} \xrightarrow{\text{3行移到第2行再相减}} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 4 & -6 & 5 & 11 \\ 0 & 1 & 5 & -8 & 8 & 14 \\ 0 & 0 & -16 & 26 & -22 & -40 \\ 0 & 0 & -18 & 28 & -24 & -44 \\ 0 & 0 & -27 & 50 & 42 & -88 \end{pmatrix} \xrightarrow{\substack{(3)-(4)\text{后再乘}\frac{1}{2} \\ (5)\times(4)\text{的2倍}}} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 4 & -6 & 5 & 11 \\ 0 & 1 & 5 & -8 & 8 & 14 \\ 0 & 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & -18 & 28 & -24 & -44 \\ 0 & 0 & 9 & -6 & 6 & 0 \end{pmatrix} \xrightarrow{(4)+(5)\text{的2倍后乘以}\frac{1}{4}\text{再用3行的相反倍加到各行}} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 1 & 3 \\ 0 & 1 & 0 & -3 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 4 & -3 & -11 \end{pmatrix} \xrightarrow{(5)\times\frac{1}{3}\text{后再乘相应倍加到各行}} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -9 \\ 0 & 1 & 0 & 0 & 0 & -14 \\ 0 & 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 13 \\ 0 & 0 & 0 & 0 & -1 & -6 \end{pmatrix}$$

$$\begin{array}{l} (1)+(4) \\ (5)+(4)\text{后再交换}(4), (5) \end{array} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & -14 \\ 0 & 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 13 \end{pmatrix} \quad \text{即得.}$$

$$D = \begin{vmatrix} 5 & 6 & 0 & 0 & 0 \\ 1 & 5 & 6 & 0 & 0 \\ 0 & 1 & 5 & 6 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 & 15 \end{vmatrix} = \frac{2^6 - 3^6}{2 - 3} = 3^6 - 2^6 = 665$$

见例2,

$$D_n = \begin{vmatrix} \alpha + \beta & \alpha\beta & & \\ 1 & \alpha + \beta & \ddots & \\ & \ddots & \ddots & \\ & & & \alpha - \beta \end{vmatrix} = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}$$

取 $\alpha = 2, \beta = 3$

且,  $D_1 = 1507$   $D_2 = -1145$   $D_3 = 703$   $D_4 = -395$   $D_5 = 212$

$$\therefore x_1 = \frac{1507}{665}, x_2 = \frac{-229}{133}, x_3 = \frac{37}{35}, x_4 = -\frac{79}{133}, x_5 = \frac{212}{665}$$

P101.20 解:

$$\begin{cases} c_0 a_1^{n-1} + c_1 a_1^{n-2} + \dots + c_{n-1} = b_1 \\ c_0 a_2^{n-1} + c_1 a_2^{n-2} + \dots + c_{n-1} = b_2 \\ \vdots \\ c_0 a_n^{n-1} + c_1 a_n^{n-2} + \dots + c_{n-1} = b_n \end{cases}$$

代入各a于f(x)

系数行列式:

$$d = \begin{vmatrix} a_1^{n-1} & a_1^{n-2} \dots a_1 & 1 \\ a_2^{n-1} & a_2^{n-2} \dots a_2 & 1 \\ \vdots & \vdots & \vdots \\ a_n^{n-1} & a_n^{n-2} \dots a_n & 1 \end{vmatrix} = \begin{vmatrix} 1 & a_1 & a_1^2 \dots a_1^{n-1} \\ 1 & a_2 & a_2^2 \dots a_2^{n-1} \\ \dots & \dots & \dots \\ 1 & a_n & a_n^2 \dots a_n^{n-1} \end{vmatrix} = (-1) C_n^2 V_n'$$

由于 $a_1, a_2 \dots a_n$ 两两不同, 故 $V_n' \neq 0, d \neq 0$ 由Cramer法则, 存在

唯一解 $C_0, C_1, C_2 \dots c_{n-1}$ , 即有 $f(x) = \sum_{i=0}^{n-1} C_i x^{n-1-i}$  (唯一地) 使 $f(a_i) = b_i$

$$\begin{cases} a_0 & = 13.60 \\ a_0 + 10a_1 + 100a_2 + 1000a_3 & = 13.57 \\ a_0 + 20a_1 + 400a_2 + 8000a_3 & = 13.55 \\ a_0 + 30a_1 + 900a_2 + 7000a_3 & = 13.52 \end{cases}$$

例P101.21 解:

$$d = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 10 & 10^2 & 10^3 \\ 1 & 20 & 20^2 & 20^3 \\ 1 & 30 & 30^2 & 30^3 \end{vmatrix} = 1.2 \times 10^7$$

$$= (1a, 20, 30 \cdot (20-10)(30-10) \cdot (30-20))$$

$$d_0 = 1.632 \times 10^8, d_1 = -50000, d_2 = 1800, d_3 = -40$$

$$a_0 = d_0/d = 13.6, a_1 = -\frac{25}{6} \times 10^{-3}, a_2 = 1.5 \times 10^{-4}, a_3 = -\frac{1}{3} \times 10^{-5}$$

$$\text{用消元法: } \begin{pmatrix} 1 & 0 & 0 & 0 & 13.6 \\ 1 & 10 & 100 & 1000 & 13.57 \\ 1 & 20 & 400 & 8000 & 13.55 \\ 1 & 30 & 900 & 27000 & 13.52 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 13.6 \\ 0 & 10^3 & 10^4 & 10^5 & -3 \\ 0 & 2 \times 10^3 & 4 \times 10^4 & 5 \times 10^5 & -5 \\ 0 & 3 \times 10^3 & 9 \times 10^4 & 27 \times 10^5 & -8 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 13.6 \\ 0 & 10^3 & 10^4 & 10^5 & -3 \\ 0 & 0 & 2 \times 10^4 & 6 \times 10^5 & 1 \\ 0 & 0 & 6 \times 10^4 & 24 \times 10^5 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 13.6 \\ 0 & 10^3 & 10^4 & 10^5 & -3 \\ 0 & 0 & 2 \times 10^4 & 6 \times 10^5 & 1 \\ 0 & 0 & 0 & 6 \times 10^5 & -2 \end{pmatrix} \rightarrow (\text{略})$$

$$h = 13.6 - \frac{25}{6} \times 10^{-3} \times t + \frac{3}{2} \times 10^{-4} \times t^2 - \frac{1}{3} \times 10.5 \times t^3 (t = ^\circ\text{C}, h = \frac{\text{g}}{\text{cm}^3})$$

$$\text{故当 } t = 15^\circ\text{C} \text{ 时, } h = 13.56 (\text{精确}) - (\frac{\text{g}}{\text{cm}^3})$$

$$\text{当 } t = 40^\circ\text{C} \text{ 时, } h = 13.46 (\text{书上答案 } 13.48 \text{ 是错的})$$

P102 补 1

$$D_1 = \sum_{j_1 j_2 \cdots j_n} \begin{vmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_n} \\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{nj_1} & a_{nj_2} & \cdots & a_{nj_n} \end{vmatrix} \quad \text{设 } D = |a_{ij}|$$

$$\therefore D_1 = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} D = \left( \sum_{j_1 j_2 \cdots j_n \text{ 取遍}} (-1)^{\tau(j_1 j_2 \cdots j_n)} \right) D = 0$$

( $n \geq 2$  奇偶排列各半)

当  $n=1$  时,



$$P102 \quad \text{证} \because D = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} a_{1j_1}(t) a_{2j_2}(t) \cdots a_{nj_n}(t)$$

$$\begin{aligned} \therefore \frac{d}{dt} D &= \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} \left( \sum_{k=1}^n a_{1j_1}(t) \cdots a_{k-1j_{k-1}}(t) \frac{da_{kj_k}(t)}{dt} a_{k+1j_{k+1}} \cdots a_{nj_n}(t) \right) \\ &= \sum_{k=1}^n \left( \sum_{j_1, j_2, \cdots, j_n} (-1)^{\tau(j_1, j_2, \cdots, j_n)} a_{1j_1}(t) a_{2j_2}(t) \cdots a_{k-1j_{k-1}}(t) \frac{da_{kj_k}(t)}{dt} a_{k+1j_{k+1}} \cdots a_{nj_n}(t) \right) \\ &= \sum_{k=1}^n \begin{vmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d}{dt} a_{k1}(t) & \frac{d}{dt} a_{k2}(t) & \cdots & \frac{d}{dt} a_{kn}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{vmatrix} \end{aligned}$$

$$\text{同理也有 } \frac{d}{dt} D = \frac{d}{dt} D^T = \begin{vmatrix} a_{11}(t) & a_{21}(t) & \cdots & a_{n1}(t) \\ \frac{d}{dt} a_{1k}(t) & \frac{d}{dt} a_{2k}(t) & \cdots & \frac{d}{dt} a_{nk}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}(t) & a_{2n}(t) & \cdots & a_{nn}(t) \end{vmatrix}$$

$$(\text{转置}) = \sum_{k=1}^n \begin{vmatrix} a_{11} & \frac{d}{dt} a_{1k}(t) & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{21} & \frac{d}{dt} a_{2k}(t) & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \frac{d}{dt} a_{nk}(t) & \cdots & a_{nn} \end{vmatrix}$$

$$\text{左} = \begin{vmatrix} a_{11}^{+x} & a_{12}^{+x} & \cdots & a_{1n}^{+x} \\ a_{21}^{+x} & a_{22}^{+x} & \cdots & a_{2n}^{+x} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{+x} & a_{n2}^{+x} & \cdots & a_{nn}^{+x} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & a_{11}^{+x} & \cdots & a_{1n}^{+x} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n1}^{+x} & \cdots & a_{nn}^{+x} \end{vmatrix}$$

P102 补 3 ①

$$\begin{vmatrix} 1 & -x & -x \cdots -x \\ 1 & a_{11} & a_{12} \cdots a_{1n} \\ 1 & a_{21} & a_{22} \cdots a_{2n} \\ \vdots & \vdots & \vdots \\ 1 & a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} + (-x) \sum_{i=1}^n (-1)^{1+(j+1)} \begin{vmatrix} 1 & a_{11} \cdots a_{1j-1} & a_{1j+1} \cdots a_{1n} \\ \vdots & \vdots & \vdots \\ 1 & a_{n1} \cdots a_{nj-1} & a_{nj+1} \cdots a_{nn} \end{vmatrix}$$

$$= D + X \sum_{i=1}^n (-1)^{j+1} \begin{pmatrix} a_{11} \cdots a_{1j-1} & a_{1j+1} \cdots a_{1n} \\ a_{21} \cdots a_{2j-1} & a_{2j+1} \cdots a_{2n} \\ \vdots & \vdots \\ a_{n1} \cdots a_{nj-1} & a_{nj+1} \cdots a_{nn} \end{pmatrix} (-1)^{j-1}$$

$$= D + X \sum_{j=1}^n \left( \sum_{i=1}^n 1 \cdot A_{ij} \right) = D + X \sum_{i=1}^n \sum_{j=1}^n A_{ij}$$

补 3 ②在①中令 X=1

$$\sum_{i=1}^n \sum_{j=1}^n A_{ij} = \begin{vmatrix} a_{11} + 1 & a_{12} + 1 & \cdots & a_{1n} + 1 \\ a_{21} + 1 & a_{22} + 1 & \cdots & a_{2n} + 1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + 1 & a_{n2} + 1 & \cdots & a_{nn} + 1 \end{vmatrix} - D =$$

$$\begin{vmatrix} a_{11} - a_{12} & a_{12} - a_{13} & \cdots & a_{1n-1} - a_{1n} & a_{1n} + 1 \\ a_{21} - a_{22} & a_{22} - a_{23} & \cdots & a_{2n-1} - a_{2n} & a_{2n} + 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} - a_{n2} & a_{n2} - a_{n3} & \cdots & a_{nn-1} - a_{nn} & a_{nn} + 1 \end{vmatrix} - \begin{vmatrix} a_{11} - a_{12} & a_{12} - a_{13} & \cdots & a_{1n-1} - a_{1n} & a_{1n} \\ a_{21} - a_{22} & a_{22} - a_{23} & \cdots & a_{2n-1} - a_{2n} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} - a_{n2} & a_{n2} - a_{n3} & \cdots & a_{nn-1} - a_{nn} & a_{nn} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} - a_{12} & a_{12} - a_{13} & \cdots & a_{1n-1} - a_{1n} & 1 \\ a_{21} - a_{22} & a_{22} - a_{23} & \cdots & a_{2n-1} - a_{2n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} - a_{n2} & a_{n2} - a_{n3} & \cdots & a_{nn-1} - a_{nn} & 1 \end{vmatrix}$$

$\beta$

P103 补 4②: 以第一行  $\times (-\alpha)$  加到后面各行

$$\begin{vmatrix} \lambda & \alpha & \alpha & \cdots & \alpha \\ b - \frac{\beta\lambda}{\alpha} & \alpha - \beta & 0 & 0 & 0 \\ b - \frac{\beta\lambda}{\alpha} & 0 & \alpha - \beta & 0 & 0 \\ b - \frac{\beta\lambda}{\alpha} & 0 & 0 & 0 & \alpha - \beta \end{vmatrix}$$

$$= (\alpha - \beta)^{n-1} \left( \lambda - \sum_{i=2}^n \frac{\alpha b - \beta\lambda}{\alpha - \beta} \right)$$

$$= \lambda(\alpha - \beta)^{n-1} - (n-1)(\alpha b - \beta\lambda)(\alpha - \beta)^{n-2}$$

P103 补 4、③ 见上面 4④得

$$D^n = [a(x+a)^n + a(x-a)^n] / 2a = \frac{1}{2} [(x+a)^n + (x-a)^n]$$

P103 补 4④

$$D_n = \begin{vmatrix} & & y \\ & y \\ * & \vdots \\ z & z \cdots z & x-y+y \end{vmatrix} = \begin{vmatrix} & & y \\ & y \\ * & \vdots \\ oo & \cdots o & x-y \end{vmatrix} + \begin{vmatrix} x & y & \cdots & \cdots & y \\ * \\ z & z & \cdots & \cdots & x & y \\ z & z & \cdots & \cdots & z & y \end{vmatrix}$$

$$= (x-y)D_{n-1} + \begin{vmatrix} x-z & y-x & o & \cdots & o \\ o & x-z & y-x & \cdots & o \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ o \cdots & & x-z & & o \\ z \cdots & \cdots & z & & y \end{vmatrix} = (x-y)D_{n-1} + y(x-z)^{n-1} \quad (i)$$

y与z的对称位置有  $D_n = (x-z)D_{n-1} + z(x-y)^{n-1}$  (ii)

(1)  $\times (x-z)$  - (ii)  $\times (x-y)$ : 得  $(y-z)D_n = y(x-z)^n - z(x-y)^n$

$$\therefore D_n = [y(x-z)^n - z(x-y)^n] / (y-z)$$

由 令,  $y=a, z=-a$ , 使得

P103, 补 5,  $f(x)$  是一个  $n+1$  级行列式

$$f(x) = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots 0 & x \\ 1 & 2 & 0 & 0 & \cdots 0 & x^3 \\ 1 & 3 & 3 & 0 & \cdots 0 & x^3 \\ \vdots & \vdots & \vdots & \vdots & \cdots \vdots & \vdots \\ 1 & n & c_n^2 & c_n^3 & \cdots c_n^{n-1} & x_n \\ 1 & n+1 & c_{n+1}^2 & c_{n+1}^3 & \cdots c_{n+1}^{n-1} & x^{n-1} \end{vmatrix}$$

计算  $f(x+1)$ , 由于前  $n$  列完全一样, 故以下只标出第  $n+1$  列

$$f(x+1) = \begin{vmatrix} * & x+1 \\ * & (x+1)^2 \\ * & (x+1)^3 \\ \cdots & \cdots \\ * & (x+1)^n \\ * & (x+1)^{n+1} \end{vmatrix} = \begin{vmatrix} * & x+1 \\ * & x^2 + 2x + 1 \\ * & x^3 + 3x^2 + 3x + 1 \\ \cdots & \cdots \\ * & x^n + C_n^{n-1}x^{n-1} + \cdots + C_n^1x + 1 \\ * & x^{n+1} + (n+1)x^n + \cdots + C_n^2x^2 + (n+1)x + 1 \end{vmatrix}$$

$$= \begin{vmatrix} x \\ x^2 \\ *x^3 \\ \vdots \\ x^n \\ x^{n+1} \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ *0 \\ \vdots \\ 0 \\ 0 \\ (n+1)x^n \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ *0 \\ c_n^2 x^{n-1} \\ c_{n+1}^2 x^{n-2} \end{vmatrix} + \cdots + \begin{vmatrix} 0 \\ 2x \\ *3x \\ \vdots \\ (n-1)x \\ nx \\ (n+1)x \end{vmatrix} + \begin{vmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{vmatrix}$$

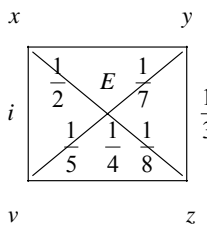
其余首项（可算出后面每个行列式的最后一列都与前面某列比例=0）

$$= f(x) + (\text{第2个行列式})D_2$$

$$\text{而 } D_2 = \begin{vmatrix} 1 \\ 2 \\ 3 \\ * & \ddots & c_n^{n-1} \\ & & (n+1)xn \end{vmatrix} = (n+1)!x^n$$

$$\therefore f(x+1) - f(x) = (n+1)!x^n$$

**P104 补 6** 分别用 U、X、Y、Z 表子该些点的电位



$$\begin{cases} (x-y)\frac{1}{2} + \frac{x}{\frac{1}{6}} + \frac{(x-0)}{1} = 100 \\ (y-x)\frac{1}{2} + \frac{x}{\frac{1}{7}} + \frac{(y-z)}{\frac{1}{3}} = 100 \\ (x-y)\frac{1}{2} + \frac{x}{\frac{1}{3}} + \frac{(x-0)}{1} = 100 \\ (0-z)\frac{1}{4} + \frac{u}{\frac{1}{5}} + \frac{(v-x)}{1} = 100 \end{cases} \begin{cases} ax - 2y - v = 100 \\ -2x + 12y - 3z = 100 \\ -3y + 15z - 4v = 100 \\ -x - 4z + 10v = 100 \end{cases}$$

$$D = \begin{vmatrix} a & -2 & 0 & -1 \\ -2 & 12 & -3 & 0 \\ 0 & -3 & 15 & 14 \\ -1 & 0 & -4 & 10 \end{vmatrix} = \begin{vmatrix} -1 & 0 & -4 & 10 \\ 0 & 12 & 5 & -20 \\ 0 & -3 & 15 & -4 \\ 0 & -2 & -36 & 89 \end{vmatrix} = \begin{vmatrix} 0 & 65 & -36 \\ -1 & +51 & -93 \\ 0 & -138 & 275 \end{vmatrix} = \begin{vmatrix} 65 & -36 \\ 138 & 275 \\ 17875 & -4968 \end{vmatrix}$$

$$dx = 210100, dy = 188400, dz = 183300, do = 223400. = 12907$$

$$\therefore x = \frac{210100}{12907}, y = \frac{188400}{12907}, z = \frac{183300}{12907}, v = \frac{223400}{12907}$$

P154,T1

1)解:

$$\begin{pmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 1 & 3 & 2 & -2 & 1 & -1 \\ 1 & -2 & 1 & -1 & -1 & 3 \\ 1 & -4 & 1 & 1 & -1 & 3 \\ 1 & 2 & 1 & -1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 5 & -4 & 0 & -1 \\ 0 & 0 & -3 & 2 & 1 & -2 \\ 0 & -5 & -4 & 3 & -1 & 2 \\ 0 & -7 & -4 & 5 & -1 & 2 \\ 0 & -1 & -4 & 3 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & -1 & -4 & 3 & 1 & -2 \\ 0 & 0 & -3 & 2 & 1 & -2 \\ 0 & 0 & 16 & -12 & -6 & 12 \\ 0 & 0 & 24 & -16 & -8 & 16 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & -1 & -4 & 3 & 1 & -2 \\ 0 & 0 & 1 & -2 & -1 & 2 \\ 0 & 0 & 0 & -4 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 5 & 0 & 0 & 1 \\ 0 & 1 & 4 & 3 & 1 & -2 \\ 0 & 0 & 1 & -2 & -1 & 2 \\ 0 & 0 & 0 & -4 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = -\frac{1}{2}k \\ x_2 = -1 - \frac{1}{2}k \\ x_3 = 0 \\ x_4 = 1 - \frac{1}{2}k \\ x_5 = k \end{cases}$$

∴方程组的解是

k为任意数

2) 解:

$$\begin{pmatrix} 1 & 2 & 0 & -3 & 2 & 1 \\ 1 & -1 & -3 & 1 & -3 & 2 \\ 2 & -3 & 4 & -5 & 2 & 7 \\ 9 & -9 & 6 & -16 & 2 & 25 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -3 & 1 & -3 & 2 \\ 0 & 3 & 3 & -4 & 5 & -1 \\ 0 & -1 & 10 & -7 & 8 & 3 \\ 0 & 0 & 33 & -25 & 29 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -3 & 1 & -3 & 2 \\ 0 & -1 & 10 & -7 & 8 & 3 \\ 0 & 0 & 22 & -25 & 29 & 8 \\ 0 & 0 & 33 & -25 & 29 & 7 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -1 & -3 & 1 & -3 & 2 \\ 0 & 1 & -10 & 7 & -8 & -3 \\ 0 & 0 & 33 & -25 & 29 & 8 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

出现了 (0,0,0,0,0,-1), 无解

$$\begin{pmatrix} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ & & 3 & 0 & 1 & 1 \\ 0 & -7 & 3 & 1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 5 & -3 & 5 & -3 \\ 0 & -7 & 3 & 1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & -2 & -2 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 2 & 0 & 12 \\ 0 & 0 & -4 & 8 & -24 \end{pmatrix}$$

3) 解:

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & -2 & -8 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 8 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -8 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{有唯一解: } x_1=-8, x_2=3, x_3=6, x_4=0$$

$$4) \text{ 解: } \begin{pmatrix} 3 & 4 & -5 & 7 & 0 \\ 2 & -3 & 3 & -2 & 0 \\ 4 & 11 & -13 & 16 & 0 \\ 7 & -2 & 1 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 7 & -8 & 9 \\ 2 & -3 & 3 & -2 \\ 0 & 17 & -19 & 20 \\ -1 & -24 & 27 & -29 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 7 & -8 & 9 \\ 0 & -17 & 19 & -20 \\ 0 & 17 & -19 & 20 \\ 0 & -17 & 19 & -20 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 7 & -8 & 9 \\ 0 & -1 & \frac{19}{17} & -\frac{20}{17} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 7 & -\frac{3}{17} & \frac{13}{17} \\ 0 & -1 & \frac{19}{17} & -\frac{20}{17} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = \frac{3}{17}k - \frac{13}{17}l \\ x_2 = \frac{19}{17}k - \frac{20}{17}l \\ x_3 = k \\ x_4 = l \end{cases}$$

得解:

5) 解:

$$\begin{pmatrix} 2 & 1 & -1 & 1 & 1 \\ 3 & -2 & 2 & -3 & 2 \\ 5 & 1 & -1 & 2 & -1 \\ 2 & -1 & 1 & -3 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & -1 & 1 & 1 \\ 1 & -3 & 3 & -4 & 1 \\ 1 & -1 & 1 & 0 & -3 \\ 0 & -2 & 2 & 4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & -3 \\ 0 & -3 & -3 & 1 & 7 \\ 0 & -1 & 2 & -4 & 4 \\ 0 & -2 & 2 & -4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 & 0 & -3 \\ 0 & -3 & -3 & 1 & 7 \\ 0 & -2 & 2 & -4 & 4 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

出现了 (0,0,0,0,-1), 无解

6) 解:

$$\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ 3 & 2 & 1 & -1 & 1 \\ 2 & 3 & 1 & -1 & 1 \\ 2 & 2 & 2 & -1 & 1 \\ 5 & 5 & 2 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ 0 & -4 & -8 & 3 & -2 \\ 0 & -1 & -5 & 2 & -1 \\ 0 & -2 & -4 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 5 & -3 & 1 \\ 0 & 0 & 12 & -10 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -\frac{5}{6} & \frac{1}{6} \\ 0 & 1 & 0 & \frac{7}{6} & \frac{1}{6} \\ 0 & 0 & 1 & -\frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{一般解为} \begin{cases} x_1 = \frac{1}{6} + \frac{5}{6}k \\ x_2 = \frac{1}{6} - \frac{7}{6}k \\ x_3 = \frac{1}{6} + \frac{5}{6}k \\ x_4 = k \cdots \end{cases} \quad \text{或} \quad \begin{cases} x_1 = (1+5x_4)/6 \\ x_2 = (1-7x_4)/6 \\ x_3 = (1+5x_4)/6 \\ x_4 \text{任意} \end{cases}$$

P154, T2

1) 解: 设  $\beta = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4$ , 则

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + x_2 - x_3 - x_4 = 2 \\ x_1 - x_2 + x_3 - x_4 = 1 \\ x_1 - x_2 - x_3 + x_4 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 5/4 \\ x_2 = 1/4 \\ x_3 = -1/4 \\ x_4 = -1/4 \end{cases}$$

$$\therefore \alpha = \frac{5}{4}\alpha_1 + \frac{1}{4}\alpha_2 - \frac{1}{4}\alpha_3 - \frac{1}{4}\alpha_4$$

2) 解: 设  $\beta = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4$ , 则

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ 0 + 3x_2 - x_4 = 0 \\ x_1 + x_2 - x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \\ x_4 = 0 \end{cases} \quad \text{即} \beta = \alpha_1 - \alpha_3$$

P155.T3

证明: 设  $k_1, k_2, \dots, k_r, l$  不全为 0, 使  $k_1\alpha_1 + k_2\alpha_2 + \dots + k_r\alpha_r + l\beta = 0$

若  $l = 0$ , 则  $k_1, \dots, k_r$  也不全为 0, 而  $k_1\alpha_1 + \dots + k_r\alpha_r = 0$  矛盾.

$$\therefore l \neq 0, \text{ 即 } \beta = \left(-\frac{k_1}{l}\right)\alpha_1 + \left(-\frac{k_2}{l}\right)\alpha_2 + \dots + \left(-\frac{k_r}{l}\right)\alpha_r \text{ 线性表出}$$

P155.T4

证明: 设  $x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n = 0$ , 则

因为系数行列式  $| (a_{ij})' | = | a_{ij} | \neq 0$ , 故上面方程组只有零解, 于是  $\alpha_1, \alpha_2, \cdots, \alpha_n$  线性无关。

证明：添加 $t_{r+1}, \dots, t_n$ , 使 $t_1, t_2, \dots, t_r, t_{r+1}, \dots, t_n$ 两两不同

由于 $\alpha_1, \alpha_2, \dots, \alpha_n$ 的分量作成Vandermonder行列式(公比两两不同)且不等于0, 由上一题,  $\alpha_1, \alpha_2, \dots, \alpha_r, \dots, \alpha_n$ 线性无关, 于是它的任一部分线性无关

而若 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 线性无关, 则 $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_4 + \alpha_1$ 线性相关。

证明: 设  $\alpha_1, \alpha_2, \dots, \alpha_s(I)$  的一个极大无关组  $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}(I)'$  及任一线性无关向量组  $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}(I)''$



任取 (I) 中的一个向量  $\beta$  有  $\alpha_{j1}, \dots, \alpha_{jr}, \beta \leftarrow (I) \Leftrightarrow (I)'$  而  $(I)'$  中只有  $r$  个向量, 由定理 2,  $\alpha_{j1}, \alpha_{jr}, \beta$  线性相关, 而本来  $(I)''$  线性无关, 故 (临界定理)  $\beta \leftarrow (I)''$ , 所以  $(I) \leftarrow (I)''$  所以  $(I)''$  是极大无关组。

P155.T8

证明:

设  $\alpha_1, \alpha_2, \dots, \alpha_s \in (I)$ ,  $\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ir} \in (I)'$ , 及  $(I)$  的一个极大无关组  $\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ir} \in (I)''$ , 已知  $(I) \leftarrow (I)'$ , 故有

$$(I)' \Leftrightarrow (I) \Leftrightarrow (I)''$$

所以取  $(I)'$  的极大无关组  $\alpha_{k1}, \alpha_{k2}, \dots, \alpha_{kt} \in (I)''$ , 则  $t \leq r$  且  $(I)'' \Leftrightarrow (I)' \Leftrightarrow (I)''$ , 那么  $(I)''' \rightarrow (I)''$  由于  $(I)''$  有  $r$  个向量且线性无关, 所以 (由定理 2 推论 1)  $r \leq t$ , 即  $r=t$ , 故  $(I)''' = (I)'$ ,  $(I)'$  线性无关。

$(I)'$  是  $(I)$  的一个极大无关组。

P155.T9

证明: 设  $(I)$  的一个线性无关组  $(I)'$

1° 逐个检查  $(I)$  中的向量  $\alpha_i$

2° a、若  $\alpha_i \leftarrow (I)'$ , 则去掉  $\alpha_i$ , 检查下一个  $\alpha$

b、若存在  $\alpha_i \not\leftarrow (I)'$ , 则添加  $\alpha_i$  到  $(I)'$  中将  $(I)'$  扩充为  $(I)''$ , 回到检查第 1 个向量, 重复 1°、2°

若干步后 ( $\because$  有限步后, 任意  $n+1$  个  $n$  维向量也相关, 必含停止), 得到  $(I)', (I)'', \dots, (I)^{(k-1)}, (I)^{(k)}$

而  $(I)^{(k)}$  不得再扩大, 于是  $(I)^{(k)}$  是一个极大无关组, 是  $(I)' \subseteq (I)^{(k)}$ 。

P155.T10

1) 解:  $\because \alpha_1$  与  $\alpha_2$  的分量不成比例, 故  $\alpha_1$  与  $\alpha_2$  线性无关

2) 解: 考虑  $\alpha_1, \alpha_2, \alpha_3 \quad \because 3\alpha_1 + \alpha_2 = \alpha_3$  去掉  $\alpha_3$

考虑  $\alpha_1, \alpha_2, \alpha_4$ , 取它们的后三个分量  $\begin{vmatrix} -1 & 2 & 4 \\ 3 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = 28 \neq 0$ ,  $\therefore$  增加一个分量后仍然线性无关。

即 $\alpha_1, \alpha_2, \alpha_4$ 线性无关

再考虑 $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ , 因为分量行列式

$$\begin{vmatrix} 1 & -1 & 2 & 4 \\ 0 & 3 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 5 & 6 \end{vmatrix} = 0$$

即 $\alpha_5 = \alpha_1 + \alpha_2 + \alpha_4$  所以它的极大线性无关组是 $\alpha_1, \alpha_2, \alpha_4$

P155.T11

1) 解:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 1 & -1 & 2 \\ 1 & 0 & 2 & 3 & -4 \\ 1 & 4 & -9 & -16 & 22 \\ 7 & 1 & 0 & -1 & 3 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 3 & -4 \\ 0 & 4 & -11 & -19 & 26 \\ 0 & 4 & -11 & -19 & 26 \\ 0 & 1 & -14 & -22 & 31 \end{pmatrix} \begin{matrix} 2 \\ 1 \\ 3 \\ 4 \end{matrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 3 & -4 \\ 0 & 1 & -14 & -22 & 31 \\ 0 & 0 & 45 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} 2 \\ 4 \\ 1 \\ 3 \end{matrix}$$

$\therefore$  秩 $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = 3$ , 且 $\alpha_2, \alpha_3, \alpha_4$ 为一个极大无关组。

$\therefore$  秩(A)=5

2) 解: 略

P156.T12

证: 设 $(I)'$ 为 $(II)''$ 分别为 $(I)$ 、 $(II)$ 的极大无关组, 则有

$$(I)' \Leftrightarrow (I) \leftarrow (II) \Leftrightarrow (II)'$$

设 $(I)'$ 含 $r$ 个向量,  $(II)'$ 含七个向量, 因为 $(I)'$ 线性无关, 且 $(I)' \leftarrow (II)'$ , 所以 $r \leq t$ , 即秩(I)  $\leq$  秩(II)

P156.T13

证明: 设 $\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ir}$ 为 $\alpha_1, \alpha_2, \dots, \alpha_n$ 的极大线性无关组则得下面表示序列

$$\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ir} \Leftrightarrow \alpha_1, \alpha_2, \dots, \alpha_n \rightarrow \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$$

因为单位向量组 $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ 线性无关, 由(定理 2 推论), 得 $n \leq r$ 故 $\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ir}$ 为

$\alpha_1, \alpha_2, \dots, \alpha_n$ 本身, 即证得 $\alpha_1, \alpha_2, \dots, \alpha_n$ 线性无关。

P156.T14

证明: 略

P156.T15

证明：“ $\Leftarrow$ ”若系数行列式 $|a_{ij}| \neq 0$ ，则由Cramer法则，对任何常数 $b_1, b_2, \dots, b_n$ 有唯一解。

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{12} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

“ $\Rightarrow$ ” 则原方程组为  $x_1\beta_1 + x_2\beta_2 + \cdots + x_n\beta_n = b$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

其中 $\beta_1, \beta_2, \dots, \beta_n$ 为A的列向量组， $\therefore$ 对任何

依次定  $b = \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ ，则得

$$\beta_1, \beta_2, \dots, \beta_n \rightarrow \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$$

从而 $\beta_1, \beta_2, \dots, \beta_n$ 线性无关，列秩(A)=n，即秩(A)=n，由定理5， $|A| \neq 0$

P156.T16

证明：设 $\alpha_1, \alpha_2, \dots, \alpha_r$ (I)及 $\alpha_1, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_s$ (II)，且秩(I)=秩(II)=t，设 $\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{it}$ (III)为(I)

的极大无在组 $\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{it}$ (IV)为(II)的极大无关组，那么， $(III) \rightleftharpoons (I) \leftarrow (II) \rightleftharpoons (IV)$

任取(II)中一个向量 $\beta$ ，组成 $\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{it}, \beta$ —(V)，则 $(V) \leftarrow (II) \leftarrow (IV)$ ，因为(IV)只有七个向量，所以(V)线性相关，而(III)线性无关。

所以 $\beta \leftarrow (III)$ 即 $(II) \leftarrow (III) \therefore (II) \rightleftharpoons (III)$

$\therefore (I) \rightleftharpoons (III) \rightleftharpoons (II)$  (I) 与 (II) 等价

P156.T17

证明： $\therefore \beta_1 = \alpha_2 + \dots + \alpha_r, \beta_2 = \alpha_1 + \alpha_3 + \dots + \alpha_r, \dots, \beta_r = \alpha_1 + \dots + \alpha_{r-1}$

$$\therefore \beta_1, \beta_2, \dots, \beta_r \rightarrow \alpha_1, \alpha_2, \dots, \alpha_r$$

$$\text{令 } r = \beta_1 + \beta_2 + \dots + \beta_r = (r-1)(\alpha_1 + \alpha_2 + \dots + \alpha_r)$$

$$\therefore \alpha_i = \frac{1}{r-1} r - \beta_i \quad \text{即 } \alpha_i \text{ 可以 } \beta_1, \beta_2, \dots, \beta_r \text{ 线性表示}$$

$$\therefore \beta_1, \beta_2, \dots, \beta_r \rightleftharpoons \alpha_1, \alpha_2, \dots, \alpha_r \quad \text{秩必相同}$$

P156.T18

$$\textcircled{1} \quad A \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -4 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & -4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \therefore \text{秩}(A)=4$$

$$\textcircled{2} \quad A \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 3 & 0 & -4 & 1 \\ 0 & 3 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \therefore \text{秩}(A)=3$$

$$\textcircled{3} \quad A \rightarrow \begin{pmatrix} 6 & 104 & 21 & 9 & 17 \\ 7 & 6 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B \quad \begin{array}{l} \text{显然 } B \text{ 中有 2 阶子式不等于 0} \\ \text{且所有 3 阶子式等于 0} \end{array} \quad \begin{array}{l} \therefore \text{秩}(B)=2 \\ \text{秩}(A)=2 \end{array}$$

$$\textcircled{4} \quad A \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 2 & 3 & 13 & 28 \\ 0 & 5 & 6 & 28 & 61 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 9 & 18 \\ 0 & 0 & 6 & 18 & 36 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \therefore \text{秩}(A)=3$$

$$\textcircled{5} \quad A \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \therefore \text{秩}(A)=5$$

P157. T19

$$\textcircled{1} \therefore \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = (\lambda+2)(\lambda-1)^2;$$

$$\therefore \text{当 } \lambda \neq -2 \text{ 时, 有唯一解。} \quad x_1 = \frac{-(\lambda+1)}{\lambda+2}, x_2 = \frac{1}{\lambda+2}, x_3 = \frac{(\lambda+1)^2}{\lambda+2}.$$

$\therefore$  当  $\lambda = -2$  时, 三个方程解相加, 得  $0=3$  (无解)。

$\therefore$  当  $\lambda = 1$  时, 变为一个方程  $x_1+x_2+x_3=1$  即  $x_1=1-x_2-x_3$ .  $x_2, x_3$  任取。

$$\textcircled{2} \because \text{系数行列式} \begin{vmatrix} \lambda+3 & 1 & 2 \\ \lambda & \lambda-1 & 1 \\ 3(\lambda+1) & \lambda & a+3 \end{vmatrix} = \lambda^3 - \lambda^2 = \lambda^2(\lambda-1)$$

而  $\lambda \neq 0$  且  $\lambda \neq 1$  时, 有唯一解: (用Cramer法则)

$$x_1 = \frac{\lambda^3 + 3\lambda^2 - 15\lambda + 9}{a^2(\lambda-1)}, x_2 = \frac{\lambda^3 + 12\lambda - 9}{\lambda^2(\lambda-1)}, x_3 = \frac{-4\lambda^2 + 3\lambda^2 + 12\lambda - 9}{\lambda^2(\lambda-1)}.$$

$$\text{而当 } \lambda=0 \text{ 时为 } \begin{cases} 3x_1 + x_2 + 2x_3 = 0 \\ -x_2 + x_3 = 0 \\ 3x_1 + 3x_3 = 3 \end{cases} \quad \textcircled{1}\text{式}+\textcircled{2}\text{式}-\textcircled{3}\text{式得 } 0=-3, \text{ 矛盾.}$$

$$\text{当 } \lambda=1 \text{ 时为 } \begin{cases} 4x_1 + x_2 + 2x_3 = 1 \\ x_1 + \dots + x_3 = 2 \\ 6x_1 + x_2 + 4x_3 = 3 \end{cases} \quad \textcircled{1}\text{式}+\textcircled{2}\text{式两倍}-\textcircled{3}\text{式得 } 0=2, \text{ 矛盾.}$$

$$\textcircled{3} \text{系数行列式} \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 2b & 1 \end{vmatrix} = b(1-a)$$

$$\text{当 } \lambda \neq 0 \text{ 且 } a \neq 1 \text{ 时, 有唯一解. } x_1 = \frac{1-2b}{b(1-a)}, x_2 = \frac{1}{b}, x_3 = \frac{4b-2ab-1}{b(1-a)}.$$

若  $b=0$ , 则 $\textcircled{2}\text{式}-\textcircled{3}\text{式}$ 得  $0=-1$ 。矛盾。

$$\text{若 } b \neq 0 \text{ 而 } a=1. \text{ 化为 } \begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + bx_2 + x_3 = 3 \\ x_1 + 2bx_2 + x_3 = 4 \end{cases}$$

$$\textcircled{1}-\textcircled{3} \text{得 } (1-2b)x_2=0$$

$$\textcircled{1}-\textcircled{2} \text{得 } (1-b)x_2=1$$

$$\therefore x_2 \neq 0 \text{ 义与 } (1-2b)=0 \text{ 即 } b=\frac{1}{2} \left( b \neq \frac{1}{2} \text{ 则矛盾无解} \right)$$

$$\text{此时化为 } \begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + \frac{1}{2}x_2 + x_3 = 3 \\ x_1 + x_2 + x_3 = 4 \end{cases} \text{ 即 } \begin{cases} x_1 + x_2 + x_3 = 4 \\ x_2 = 2 \end{cases} \text{ 即 } \begin{cases} x_1 = 2 - x_3 \\ x_2 = 2 \end{cases} x_3 \text{ 任意值}$$

此时化为  
P157.T20、

$$1) \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 & -3 \\ 0 & 1 & 2 & 2 & 6 \\ 5 & 4 & 3 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -6 \\ 0 & 1 & 2 & -6 \\ 0 & -1 & -2 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -1 & -5 \\ 0 & 1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

即  $x_3, x_4, x_5$  为自由未知量.

$$\begin{aligned}
 (1,0,0) & \quad \eta_1 = (1,-2,1,0,0) \\
 (0,1,0) & \quad \eta_2 = (1,-2,0,1,0) \\
 \text{令 } (x_3, x_4, x_5) = (0,0,1) & \quad \text{得: } \eta_3 = (1,-2,0,0,1)
 \end{aligned}$$

2)

$$\begin{aligned}
 & \begin{pmatrix} 1 & 1 & 0 & -3 & -1 \\ 1 & -1 & 2 & -1 & 0 \\ 4 & -2 & 6 & 3 & -4 \\ 2 & 4 & -2 & 4 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & -3 & -1 \\ 0 & -2 & 2 & 2 & 1 \\ 0 & -6 & 6 & 15 & 0 \\ 0 & 2 & -2 & 10 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & -3 & -1 \\ 0 & -2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 12 & -4 \end{pmatrix} \\
 & \rightarrow \begin{pmatrix} 1 & 0 & 1 & -2 & -\frac{1}{2} \\ 0 & 1 & -1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & -\frac{7}{6} \\ 0 & 1 & -1 & 0 & -\frac{5}{6} \\ 0 & 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

即  $x_1, x_2, x_5$  为基本,  $x_3, x_4$  为自由未知量。

$$\begin{aligned}
 & \eta_1 = (-1, 1, 1, 0, 0) \\
 & (x_3, x_4) = (1, 0) \\
 & \text{令 } (x_3, x_4) = (0, 1), \text{ 得 } \eta_2 = \left(\frac{7}{6}, \frac{5}{6}, 0, \frac{1}{3}, 1\right).
 \end{aligned}$$

即基础解系为  $(-1, 1, 1, 0, -2)$  和  $(7, 5, 0, 2, 6)$

$$4) \quad \begin{pmatrix} 1 & -2 & 1 & -1 & 1 \\ 2 & 1 & -1 & 2 & -3 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -5 & 1 & -2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 & -1 & 1 \\ 0 & 5 & -3 & 4 & -5 \\ 0 & 4 & -4 & 4 & -5 \\ 0 & -1 & -1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -8 & 4 & -5 \\ 0 & 0 & -8 & 4 & -5 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & +\frac{1}{2} & -\frac{7}{8} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{5}{8} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{5}{8} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{即} \quad \begin{cases} x_1 = -\frac{1}{2}x_4 + \frac{7}{8}x_5 \\ x_2 = -\frac{1}{2}x_4 + \frac{7}{8}x_5 \\ x_3 = -\frac{1}{2}x_4 + \frac{7}{8}x_5 \end{cases}$$

$$(x_4, x_5) = (1, 0)(0, 1) \text{ 得基础解系 } \begin{cases} y_1 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 1, 0\right) \\ y_2 = \left(\frac{7}{8}, \frac{5}{8}, -\frac{5}{8}, 0, 1\right) \end{cases}$$

令

$$\begin{array}{l} \text{P157. T22} \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & -3 & a \\ 0 & 1 & 2 & 6 & 3 \\ 5 & 4 & 3 & -1 & b \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -2 & -6 & a-3 \\ 0 & 1 & 2 & 2 & 6 & 3 \\ 0 & -1 & -2 & -2 & -6 & b-5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -1 & -5 & -2 \\ 0 & 1 & 2 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & b-2 \end{pmatrix}$$

由于有解  $\Leftrightarrow$  秩(系)=秩(增) 故有解  $\Leftrightarrow a=0, b=2$ .

此时,  $x_1, x_2$  为基础未知量。特解为  $x_0=(-2, 3, 0, 0, 0)$

$x_3, x_4, x_5$  为自由未知量, 依次取

$$(x_3, x_4, x_5) = (1, 0, 0) \quad \eta_1 = (1, -2, 1, 0, 0)$$

$$(x_3, x_4, x_5) = (0, 1, 0) \quad \text{得} \quad \eta_2 = (1, -2, 0, 1, 0)$$

$$(x_3, x_4, x_5) = (0, 0, 1) \quad \eta_3 = (5, -6, 0, 0, 1)$$

通解为  $x_0 + k_1 \eta_1 + k_2 \eta_2 + k_3 \eta_3$  ( $k_1, k_2, k_3$  为任意常数。)

P158. T23

$$\text{增广矩阵} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & a_1 \\ & 1 & -1 & 0 & 0 & a_2 \\ & & -1 & 0 & a_3 \\ & & 1 & -1 & a_4 \\ -1 & 0 & 0 & 0 & 1 & a_5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ & & 1 & -1 & 0 & a_3 \\ & & & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & a_1 + a_2 + a_3 + a_4 + a_5 \end{pmatrix}$$

因为系数矩阵秩为 4。增广矩阵秩为 5  $\Leftrightarrow \sum_{i=1}^5 a_i \neq 0$

$$\text{秩为 4} \quad \Leftrightarrow \sum_{i=1}^5 a_i = 0$$

故由有解判别定理, 方程组有解  $\Leftrightarrow$  秩(系)=秩(增)  $\Leftrightarrow \sum_{i=1}^5 a_i = 0$

$$\sum_{i=1}^5 a_i = 0 \quad \text{有解时, 即} \quad \text{。矩阵化为最简阶梯} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & a_1 + a_2 + a_3 + a_4 \\ 0 & 1 & 0 & -1 & a_2 + a_3 + a_4 \\ 0 & 0 & 1 & -1 & a_3 + a_4 \\ 0 & 0 & 0 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

特解  $r_0 = (a_1 + a_2 + a_3 + a_4, a_2 + a_3 + a_4, a_3 + a_4, 0)$ 。

导出组基础系 (只有一个自由未知数  $x_5=1$ ) 为  $\eta = (1, 1, 1, 1, 1)$ 。

所以方程组的通解为  $r_0 + k\eta$ 。  $k$  为任意的数。

P158.T24

设  $\eta_1, \eta_2, \dots, \eta_s$  是某齐次方程组的基础解系, 而  $\xi_1, \xi_2, \dots, \xi_t$  是方程组的线性无关解组。则  $\eta_1, \eta_2, \dots, \eta_s \Leftrightarrow \xi_1, \xi_2, \dots, \xi_t$ 。由于等价且都线性无关, 必有  $s=t$ 。由传递性, 方程的任一解可由  $\eta_1, \eta_2, \dots, \eta_s$  线性表示, 也可由  $\xi_1, \xi_2, \dots, \xi_t$  线性表示。  $\xi_1, \xi_2, \dots, \xi_t$  也是基础解系。

P158.T25

由于秩 (系)  $=r$ , 故基础解系含  $n-r$  个向量  $\eta_1, \eta_2, \dots, \eta_{n-r}$  ( $r < n$ )

而它们任意  $n-r$  解  $\zeta_1, \zeta_2, \dots, \zeta_{n-r}$ 。如果线性无关。则由 (补 6, P157) 的证明方法: (见

3.41.6.\*)。秩  $(\eta_1, \eta_2, \dots, \eta_{n-r}) = \text{秩}(\zeta_1, \zeta_2, \dots, \zeta_{n-r}) = n-r$ 。且  $\zeta_1, \zeta_2, \dots, \zeta_{n-r} \leftarrow \eta_1, \eta_2, \dots, \eta_{n-r}$ 。故,  $\zeta_1, \zeta_2, \dots, \zeta_{n-r} \Leftrightarrow \eta_1, \eta_2, \dots, \eta_{n-r}$  (再由上题 (24 题) 知向量  $\zeta_1, \zeta_2, \dots,$

$\zeta_{n-r}$  也为方程组的基础解系。

P158.T26

证明: 设  $\eta_v = (k_{v1}, k_{v2}, \dots, k_{vn})$  ( $v=1, 2, \dots, t$ ) 为方程组  $\sum_{j=1}^n a_{ij}x_j = b_j (j=1, 2, \dots, s)$  的解,

即必有  $\sum_{j=1}^n a_{ij}k_{vj} = b_j \quad \begin{pmatrix} v=1 \dots t \\ i=1 \dots s \end{pmatrix}$ , 那么  $\eta = \sum_{v=1}^n u_v \eta_v$  代入方程组得

$$\sum_{j=1}^n a_{ij} \sum_{v=1}^t u_v k_{vj} = \sum_{v=1}^t \left( \sum_{j=1}^n a_{ij} k_{vj} \right) u_v = \sum_{v=1}^t b_j u_v = b_i \sum_{v=1}^t u_v = b_i \quad \left( \text{由已知条件 } \sum_{v=1}^t u_v = 1 \right)$$

$\therefore \eta = \sum_{v=1}^n u_v \eta_v$  是方程的解。

此题反过来也成立, 即

若  $\sum_{v=1}^t u_v \eta_v$  为非齐次方程的解, 且  $\eta_v$  也是解, 则必有  $\sum_{v=1}^t u_v = 1$ 。

P158.T27



$$\begin{vmatrix} a_0, a_1 \cdots a_n \\ a_0 \cdots a_n \\ a_0 \cdots a_n \\ b_0 \cdots b_m \\ b_0 \cdots b_m \\ b_0 \cdots b_m \end{vmatrix} = (-1)^m \begin{vmatrix} b_0, b_1 \cdots b_n \\ a_0, a_1 \cdots a_n \\ a_0 \cdots a_n \\ a_0 \cdots a_n \\ b_0 \cdots b_m \\ b_0 \cdots b_m \end{vmatrix} = (-1)^m \cdots = (-1)^{mn} \begin{vmatrix} b_0, b_r \cdots b_m \\ b_0 \cdots b_m \\ a_0 \cdots a_n \\ a_0 \cdots a_m \end{vmatrix} = (-1)^{mn} R(g.f)$$

$$R(f \cdot g) = \text{其中 } n = \partial(f(x)) \quad m = \partial(g(x))$$

158. T28

①

$$R(f \cdot g) = \begin{vmatrix} 5 & -6x & 5x^2 - 16 & 0 \\ 0 & 5 & -6x & 5x^2 - 16 \\ 1 & -x - 1 & 2x^2 - x - 4 & 0 \\ 0 & 1 & -x - 1 & 2x^2 - x - 4 \end{vmatrix} = \begin{vmatrix} 0 & -x + 5 & 0 \\ 0 & 0 & -5x^2 + 5x + 4 \\ 1 & -x - 1 & \\ 0 & 1 & 2x^2 - x - 4 \end{vmatrix}$$

$$= (-1)^{3+1} \begin{vmatrix} 0 & -6x^2 + 9x + 9 & (2x^2 - x - 4) \\ 0 & -x + 5 & -5x^2 + 5x + 4 \\ 1 & -x - 1 & 2x^2 - x - 4 \end{vmatrix} \quad \text{直接展开方程相加}$$

$$= 32x^4 - 96x^3 + 96x^2 - 64x = 32(x^4 - 3x^3 + x^2 + 3x - 2) = 32(x^2 - 1)(x^2 - 3x + 2)$$

$$= 32(x - 1)^2(x + 1)(x - 2)$$

有 4 个解是  $x_1 = x_2 = 1$ ,  $x_3 = 2$ ,  $x_4 = -1$ 。

$$\text{用 } x=1 \text{ 代入在方程组得 } \begin{cases} 5y^2 - 6y - 11 = 0 \\ y^2 - 2y - 3 = 0 \end{cases} \text{ 有公共解 } y = -1, \text{ 即 } \begin{cases} x = 1 \\ y = -1 \end{cases}$$

$$\text{用 } x=2 \text{ 代入在方程组得 } \begin{cases} 5y^2 - 12y + 4 = 0 \\ y^2 - 3y - 3 = 0 \end{cases} \text{ 有公共解 } y = 2, \text{ 即 } \begin{cases} x = 2 \\ y = 2 \end{cases}$$

$$\text{用 } x=-1 \text{ 代入在方程组得 } \begin{cases} 5y^2 + 6y - 11 = 0 \\ y^2 - 1 = 0 \end{cases} \text{ 有公共解 } y = 1, \text{ 即 } \begin{cases} x = -1 \\ y = 1 \end{cases}$$

$$\text{即得到三组解 } \begin{cases} x = 1 \\ y = -1 \end{cases} \begin{cases} x = 2 \\ y = 2 \end{cases} \begin{cases} x = -1 \\ y = 1 \end{cases}$$

#### 第四章 矩阵练习题参考答案

$$\textcircled{1}\text{解: } AB = \begin{pmatrix} 6 & 2 & -2 \\ 6 & 1 & 0 \\ 8 & -1 & 2 \end{pmatrix} \quad BA = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 3 & 4 \end{pmatrix}$$

$$AB - BA = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 0 & 0 \\ 4 & -4 & -2 \end{pmatrix}$$

$\therefore$

$$\textcircled{2}\text{解: } AB = \begin{pmatrix} a+b+c & a^2+b^2+c^2 & ac+b^2+ac \\ a+b+c & ac+b^2+ac & a^2+b^2+c^2 \\ 3 & a+b+c & a+b+c \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & a & c \\ 1 & b & b \\ 1 & c & a \end{pmatrix} \begin{pmatrix} a & b & c \\ c & b & a \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a+ac+c & b+ab+c & c+a^2+c \\ a+bc+b & b+b^2+b & c+ab+b \\ a+c^2+a & b+bc+a & c+ac+a \end{pmatrix}$$

$$\therefore AB - BA = \begin{pmatrix} b-ac & a^2+b^2+c^2-b-ab-c & b^2-a^2+2ac-2c \\ c-bc & 2(ac-b) & a^2+b^2+c^2-b-ab-c \\ 3-2a-c^2 & c-bc & b-ac \end{pmatrix}$$

P198. T 2

$$\textcircled{1}\text{解: } \begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 12 & 5 & 5 \\ 12 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix}$$

$$\textcircled{2}\text{解: } \begin{pmatrix} 3 & 2 \\ -4 & -2 \end{pmatrix}^5 = \begin{pmatrix} 3 & 2 \\ -4 & -2 \end{pmatrix}^3 \begin{pmatrix} 1 & 2 \\ -4 & -4 \end{pmatrix} = \begin{pmatrix} -5 & -2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -4 & -4 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 4 & 8 \end{pmatrix}$$

$$\textcircled{4}\text{解: } \therefore \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{pmatrix}$$

$$\therefore \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}^n = \begin{pmatrix} \cos n\varphi & -\sin n\varphi \\ \sin n\varphi & \cos n\varphi \end{pmatrix}$$

5 解:

$$(2, 3, -1) \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 2 - 3 + 1 = 0$$

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} (2, 3, -1) = \begin{pmatrix} 2 & 3 & -1 \\ -2 & -3 & 1 \\ -2 & -3 & 1 \end{pmatrix}$$

6 解:

$$(x, y, 1) \begin{pmatrix} a_{11}x + a_{12}y + b_1 \\ a_{12}x + a_{22}y + b_2 \\ b_1x + b_2y + c \end{pmatrix} = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2b_1x + 2b_2y + c$$

原式=

$$A = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} A^2 = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \therefore A^n = \begin{cases} 2^n E & n = 2k \text{ 时} \\ 2^{n-1} A & n = 2k + 1 \text{ 时} \end{cases}$$

7 解:

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^2 = \begin{pmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda \end{pmatrix},$$

8 解: 而

$$\begin{pmatrix} \lambda^k & c_k^1 \lambda^{k-1} & c_k^2 \lambda^{k-2} \\ 0 & \lambda^k & c_k^1 \lambda^{k-1} \\ 0 & 0 & \lambda^k \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^{k+1} & c_{k+1}^1 \lambda^k & c_{k+1}^2 \lambda^{k-1} \\ 0 & \lambda^{k+1} & c_{k+1}^1 \lambda^k \\ 0 & 0 & \lambda^{k+1} \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & c_n^1 \lambda^{n-1} & c_n^2 \lambda^{n-2} \\ 0 & \lambda^n & c_n^1 \lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{1}{2}n(n-1)\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$$

∴

P198. T3

$$A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 8 & 2 & 4 \\ 11 & 2 & 5 \\ -1 & 0 & -1 \end{pmatrix}$$

①

$$\therefore f(A) = A^2 - A - E = \begin{pmatrix} 6 & 1 & 3 \\ 8 & 0 & 3 \\ -2 & 2 & -1 \end{pmatrix} - E = \begin{pmatrix} 5 & 1 & 3 \\ 8 & -1 & 3 \\ -2 & 2 & -2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}^2 = \begin{pmatrix} 7 & -5 \\ -15 & 12 \end{pmatrix}$$

②

$$\therefore f(A) = A^2 - 5A + 3E = A^2 - \begin{pmatrix} 7 & -5 \\ -15 & 12 \end{pmatrix} = 0$$

P199. T4.

$$\textcircled{1} \text{ 设 } X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, AX = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}, XA = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix},$$

$$\text{由 } AX = XA \Rightarrow c = 0, a+b = b+d \Rightarrow a = d$$

$$\therefore X = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} a, b \text{ 任取.}$$

$$\textcircled{2} \therefore XA = AX \Leftrightarrow X(A-E) = (A-E)X. \quad \bar{A} = A - E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$

$$\text{设 } X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, X\bar{A} = \begin{pmatrix} 3x_{13} & x_{13} & 2x_{12} + x_{13} \\ 3x_{23} & x_{23} & 2x_{22} + x_{23} \\ 3x_{33} & x_{33} & 2x_{32} + x_{33} \end{pmatrix},$$

$$\bar{A}X = \begin{pmatrix} 0 & 0 & 0 \\ 2x_{31} & 2x_{32} & 2x_{33} \\ 3x_{11} + x_{21} + x_{31} & 3x_{12} + x_{22} + x_{32} & 3x_{13} + x_{23} + x_{33} \end{pmatrix}$$

$$\therefore x_{12} = x_{13} = 0$$

$$\begin{aligned} x_{21} = 3a \quad x_{31} = 3c \quad x_{11} = b + c - a - c \\ x_{22} = b \Rightarrow x_{32} = c \Rightarrow x_{22} = 0 \\ \text{令 } x_{23} = 2c \quad x_{33} = b + c \quad x_{13} = 0 \end{aligned} \quad \therefore X = \begin{pmatrix} b-a & 0 & 0 \\ 3a & b & 2c \\ 3c & c & b+c \end{pmatrix}$$

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, AX = \begin{pmatrix} x_{25} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ 0 & 0 & 0 \end{pmatrix}, XA = \begin{pmatrix} 0 & x_{11} & x_{12} \\ 0 & x_{21} & x_{22} \\ 0 & x_{31} & x_{32} \end{pmatrix}$$

③ 同样设

$$x_{21} = x_{31} = x_{32} = 0, x_{11} = x_{22} = x_{33}, x_{23} = x_{12} \therefore X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{11} & x_{12} \\ 0 & 0 & x_{11} \end{pmatrix}$$

$\therefore$

P199. T5

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & \cdots & \cdots & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \therefore AX = XA$$

左边*i*行*j*列的元为  $a_i x_{ij}$

右边*i*行*j*列的元素  $x_{ij} a_j$

$$\therefore i \neq j, \quad a_i x_{ij} = a_j x_{ij} \Rightarrow (a_i - a_j) x_{ij} \Rightarrow x_{ij} = 0 \quad (\because a_i \neq a_j)$$

$$\therefore X = \begin{pmatrix} x_{11} & & & \\ & x_{22} & & \\ & & \cdots & \\ & & & x_{nn} \end{pmatrix} \quad \text{只是对角矩阵}$$

P199. T6

$$A = \begin{pmatrix} a_1 E_{n_1} & & & \\ & a_2 E_{n_2} & & \\ & & \cdots & \\ & & & a_r E_{n_r} \end{pmatrix} \quad (n_1 + n_2 + \cdots + n_r = n)$$

$$\text{令 } X = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1r} \\ X_{21} & X_{22} & \cdots & X_{2r} \\ & \cdots & \cdots & \\ X_{r1} & X_{r2} & \cdots & X_{rr} \end{pmatrix}$$

且  $X_{ij}$  为  $n_i \times n_j$  型才能  $AX = XA$  分块相乘, 应有

$$\left. \begin{array}{l} \text{左边 } AX \text{ 第 } i \text{ 块行 } j \text{ 块列为 } a_i E_{n_i} \cdot X_{ij} = a_i X_{ij} \\ \text{右边 } XA \text{ 第 } i \text{ 块行 } j \text{ 块列为 } X_{ij} \cdot a_j E_{n_j} = a_j X_{ij} \end{array} \right\} \quad \because i \neq j, a_i \neq a_j$$

$$\therefore i \neq j \text{ 时, } x_{ij} = 0. \therefore X = \begin{pmatrix} X_{11} & \cdots & \\ & X_{22} & \cdots \\ & \cdots & \cdots \\ & & \cdots & X_{rr} \end{pmatrix} \quad \text{为与 } A \text{ 同类型的准对角矩阵}$$

P199. T7

$$\text{① 设 } A = (a_{ij})_{n \times n}. \quad AE_{12} = \begin{pmatrix} 0 & a_{11} & \cdots & 0 \\ 0 & a_{21} & \cdots & 0 \\ 0 & a_{31} & \cdots & 0 \\ 0 & \vdots & \cdots & 0 \\ 0 & a_n & \cdots & 0 \end{pmatrix}, \quad E_{12}A = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

$\therefore A$  的第一列  $a_{11} = a_{22}$ , 其余  $a_{k1} = 0 (k > 1)$

$$A = \begin{pmatrix} a_{11} & & * & & \\ 0 & a_{11} & 0 & \cdots & \cdots & 0 \\ 0 & & & & & \\ \vdots & & * & & & \\ 0 & & & & & \end{pmatrix}$$

$A$  的第二行  $a_{22} = a_{11}$ , 其余  $a_{2s} = 0 (s \neq 2)$

$$AE_{ij} = \begin{pmatrix} & a_{1i} \\ & a_{2i} \\ 0 & \vdots & 0 \\ & a_{ni} \end{pmatrix}, E_{ij}A = \begin{pmatrix} 0 & 0 & 0 \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ 0 & 0 & 0 \end{pmatrix} \text{第 } i \text{ 行}$$

②

∴ A 的第 i 列:  $a_{ii} = a_{jj}$ , 且  $a_{ki} = 0$ , ( $k \neq i$ )

A 的第 j 行,  $a_{ji} = a_{ii}$  且  $a_{js} = 0$ , ( $s \neq j$ )

③ 由于 A 与所有 n 级矩阵可换, 故 A 与  $E_{11}, E_{12}, E_{13} \cdots E_{1n}$  可换

∴ A 的第一列全为 0,  $AE_{11} = E_{11}A \Rightarrow A$  的第一行只留下  $a_{11}$  可解非 0

$AE_{12} = E_{12}A \Rightarrow A$  的第二行只留下  $a_{22}=a_{11}$  其余全为 0

$AE_{13} = E_{13}A \Rightarrow A$  的第三行只留下  $a_{33}=a_{11}$ , 其余全为 0

$AE_{1n} = E_{1n}A \Rightarrow A$  的第 n 行只留下  $a_{nn}=a_{11}$ . 其余全为 0

$$A = \begin{pmatrix} a_{11} & & & 0 \\ & a_{11} & & \\ & & a_{11} & \\ 0 & & & a_{11} \end{pmatrix} = aE \quad (a = a_{11})$$

所以

P200. T8

$$A(B+C) = AB + AC = BA + CA = (B+C)A$$

$$A(BC) = (AB)C = (BA)C = B(AC) = BC(A) = BC(A) = (BC)A$$

P200. T9

$$" \Rightarrow " \text{ 若 } A^2 = A, \text{ 则 } \frac{1}{4}(B^2 + 2B + E) = \frac{1}{2}(B + E) \Rightarrow \frac{1}{4}B^2 - \frac{1}{4}E = 0 \text{ 得 } B^2 = E$$

$$" \Leftarrow " \text{ 若 } B^2 = E, \text{ 则 } A^2 = \frac{1}{4}(B^2 + 2B + E) = \frac{1}{4}(E + 2B + E) = \frac{1}{2}(B + E) = A$$

P200. T10

反设  $A \neq 0$ , 不妨设  $a_{st} \neq 0$ , 那么  $a_{ts} \neq 0$ , 那么  $A^2$  中第 s 行 s 列的元素 为

$$\sum_{k=1}^n a_{sk}a_{ks} = \sum_{k=1}^n a_{sk}a_{sk} = a_{s1}^2 + a_{s2}^2 + a_{st}^2 + \cdots + a_{sn}^2 > 0.$$

∴  $A^2 \neq 0$ , 矛盾, 即  $A = 0$ 。

P200. T11

$$" \Rightarrow "(AB)' = AB \Rightarrow AB = (AB)' = B'A' = BA (\because B' = B, A' = A)$$

"  $\Leftarrow$  " 如果  $AB = BA$ , 那么  $(AB)' = B'A' = BA = AB$ , 为对称矩阵。

P200. T12

设  $A=B+C$ ,  $(B'=B, C'=-C)$

$$\therefore A' = B' + C' = B - C \quad \therefore B = \frac{1}{2}(A + A'), C = \frac{1}{2}(A - A')$$

恰如  $B' = B, C' = -C$ , 即为所求

P200. T13

$$D = \begin{pmatrix} 1 & 1 & & 1 \\ x_1 & x_2 & \vdots & x_n \\ x_1^2 & x_2^2 & \vdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & & x_n^{n-1} \end{pmatrix}, D' = \begin{pmatrix} 1 & x_1 & x_1^2 & & x_1^{n-1} \\ 1 & x_2 & x_2^2 & & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & & x_n^{n-1} \end{pmatrix}$$

$$\begin{aligned} \text{令 } DD' &= (a_{ij})_{n \times n} = A, a_{ij} = \sum_{k=1}^n x_k^{i-1+j-1} = s_{i+j-2} \\ \therefore |A| &= |(a_{ij})| = |DD'| = |D|^2 = \prod_{1 \leq i < j \leq n} (x_j - x_i)^2 \\ \therefore \end{aligned}$$

P200. T14

" $\Rightarrow$ " 取  $B_i$  为  $B$  的一个非 0 列  $\therefore AB_i = 0$ , 而  $AX = 0$  有非零解. 故  $|A| = 0$ .

" $\Leftarrow$ "  $\therefore |A| = 0 \therefore AX = 0$  有非零解  $x_0 \neq 0$ , 令  $B_1 = x_0, B_2 = 2x_0, \dots, B_n = nx_0$

而  $B$  的列由  $B_1, B_2, \dots, B_n$  组成, 所以  $B \neq 0$

P200. T15

考虑  $AE$ .  $\because E$  的每一列  $E_i$  去乘  $A$  的各行为 0,  $\therefore AE = 0$

又  $AE = A \therefore A = 0$

P200. T16.

① 考虑齐线方程组,  $C'x = 0 (\because C'_{n \times r})$  只含  $r$  个未知量,

而秩  $(C') = \text{秩}(C) = r = \text{未知量个数} \therefore C'X = 0$  只有零解

$\therefore BC = 0 \Rightarrow C'B' = 0' = 0 \therefore B'$  的各列 (都是适合  $C'X = 0$ ) 都为 0

$\therefore B' = 0, B = 0$

② 若  $BC = C \Rightarrow (B - E)C = 0 \Rightarrow B - E = 0 \Rightarrow B = E$

P200. T17

设  $A$  的行向量为  $\alpha_1, \alpha_2, \dots, \alpha_s$ , (I),  $B$  的行向量为  $\beta_1, \beta_2, \dots, \beta_s$  (II), 而

$C = A + B$  的行向量为  $\gamma_1, \gamma_2, \dots, \gamma_s$ , (III)。那么

$$r_1 = \alpha_1 + \beta_1, r_2 = \alpha_2 + \beta_2 \cdots, r_m = \alpha_m + \beta_m。$$

∴ 设  $\alpha_{i1}, \cdots, \alpha_{ir} (I)'$  为 (I) 的极大无关组, 那么秩(A) = 秩(I) = r

设  $\beta_{j1}, \cdots, \beta_{jp} (II)'$  为 (II) 的极大无关组, 那么秩(A) = 秩(II) = p

$$\therefore (III) \leftarrow (I) \cup (II) \leftarrow (I)' \cup (II)' = \{\alpha_{i1}, \cdots, \alpha_{ir}, \beta_{j1}, \cdots, \beta_{jp}\} \cdots (IV)$$

$$\therefore \text{秩}(A+B) = \text{秩}(C) = \text{秩}(III) \leq \text{秩}(IV) \leq r+p = \text{秩}(A) + \text{秩}(B)。$$

P200. T18

设秩(A) = r, 那么, 线性方程组 AX=0 的基础解系可设为  $\eta_1, \eta_2 \cdots \eta_{n-r}$ 。

设B的各列为  $B_1, B_2, \cdots, B_n$  ∴ AB=0. 说明B的每列  $B_j$  乘以A的每行都为 0, 即时  $B_j$  是 AX=0 的解。∴

$$B_j \leftarrow \eta_1, \eta_2 \cdots \eta_{n-r}$$

$$\therefore B_1, B_2 \cdots B_n, \leftarrow \eta_1, \eta_2 \cdots \eta_{n-r}$$

$$\therefore \text{秩}(B) = \text{秩}(B_1, B_2 \cdots B_n) \leq \text{秩}(\eta_1, \eta_2 \cdots \eta_{n-r}) = n-r$$

$$\therefore \text{秩}(A) + \text{秩}(B) \leq r + n - r = n$$

P200. T19

若  $A^k=0$

$$\therefore (E-A)(E+A+A^2+\cdots+A^{k-1}) = E+A+A^2+\cdots+A^{k-1} - A - A^2 - \cdots - A^{k-1} - A^k$$

$$= E - A^k = E - 0 = E$$

$$\therefore (E-A)^{-1} = E + A + A^2 + \cdots + A^{k-1}$$

P201. T20

$$\textcircled{1} A^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, |A|=1 \quad \therefore A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\textcircled{2} A = \left( \begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 0 \end{array} \right) = \begin{pmatrix} A_1 & A_2 \\ A_3 & 0 \end{pmatrix}, \quad \text{令 } A^{-1} = X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

$$\therefore AX = \begin{pmatrix} A_1 X_1 + A_2 X_3 & A_1 X_2 + A_2 X_4 \\ A_3 X_1 & A_3 X_2 \end{pmatrix}$$

$$A_3 X_1 = 0 \Rightarrow \because A_3^{-1} \text{存在} \therefore X_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A_3 X_2 = E_2 \Rightarrow X_2 = A_3^{-1} = -\frac{1}{3} \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$A_1 X_1 + A_2 X_3 = E_1 \Rightarrow A_2 X_3 = E_1 \Rightarrow X_3 = -1$$



$$\text{而 } A_1 X_2 + A_2 X_4 = 0 \Rightarrow X_4 = -A_2^{-1} A_1 X_2$$

$$X_4 = -(-1)(1,1) \begin{pmatrix} -\frac{2}{3} \\ -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \quad \therefore A^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\text{③ } |A| = \begin{vmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{vmatrix} = -2 + 0 + 6 - 3 - 0 - 2 = -1 \quad \text{即 } \begin{matrix} A_{11} = -1 & A_{21} = 4 & A_{31} = 3 \\ A_{12} = -1 & A_{22} = 5 & A_{32} = 3 \\ A_{13} = 1 & A_{23} = -6 & A_{33} = -4 \end{matrix}$$

$$\therefore A^* = \begin{pmatrix} -1 & 4 & 3 \\ -1 & 5 & 3 \\ 1 & -6 & -4 \end{pmatrix} \quad A^{-1} = |A|^{-1} A^* = \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}$$

$$\text{④ } A = \left( \begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 2 \\ \hline 3 & 1 & 1 & -1 \\ 1 & 0 & -2 & -6 \end{array} \right) \quad A_1 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad A_2 = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 1 & -1 \\ -2 & -6 \end{pmatrix} \quad A_1^{-1} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} E & O \\ -A_3 A_1^{-1} & E \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 - A_3 A_1^{-1} A_2 \end{pmatrix} \quad A_3 A_1^{-1} = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}$$

$$A_4 - A_3 A_1^{-1} A_2 = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = B_4$$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}^{-1} = \begin{pmatrix} A_1 & A_2 \\ O & B_4 \end{pmatrix}^{-1} \begin{pmatrix} E & O \\ -A_3 A_1^{-1} & E \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 5 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{pmatrix}$$

$$\therefore$$

$$A^{-1} = \begin{pmatrix} -3 & 2 & -26 & 17 \\ 2 & -1 & 20 & -13 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 22 & -6 & -26 & 17 \\ -17 & 5 & 20 & -13 \\ -1 & 0 & 2 & -1 \\ 4 & -1 & -5 & 3 \end{pmatrix}$$

$$\text{5 法 1: } A^2 = 4E \therefore A^{-1} = \frac{1}{4} A$$

$$\text{法 2: } \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 0 & -1 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{cccc|cccc} 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & -1 & 0 & 0 & -1 \\ 0 & -2 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -2 & -1 & 1 & 0 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 2 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & -2 & 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & -2 & -1 & 1 & 0 & 0 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{cccc|cccc} 2 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -4 & -1 & 1 & 1 & -1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 4 & 0 & 0 & 4 & 2 & 0 & 0 & 2 \\ 0 & 4 & 0 & 4 & 2 & 0 & -2 & 0 \\ 0 & 0 & 4 & -4 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 4 & 1 & -1 & -1 & 1 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{cccc|cccc} 4 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 4 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 4 & 1 & -1 & -1 & 1 \end{array} \right) \Rightarrow \left( E \mid \frac{1}{4} ( \quad ) \right)$$

$$\therefore A^{-1} = \frac{1}{4} ( \quad ) = \frac{1}{4} A$$

$$\text{⑥ } A = \begin{pmatrix} 3 & 3 & -4 & -3 \\ 0 & 6 & 1 & 1 \\ 5 & 4 & 2 & 1 \\ 2 & 3 & 3 & 2 \end{pmatrix} (A, E) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & -7 & -5 & 1 & 0 & 0 & -1 \\ 0 & 6 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 37 & 26 & -5 & 0 & 1 & 5 \\ 0 & 3 & 17 & 12 & -2 & 0 & 0 & 3 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & -7 & -5 & 1 & 0 & 0 & -1 \\ 0 & 1 & 20 & 14 & -3 & 0 & 1 & 2 \\ 0 & 0 & -33 & -23 & 4 & 1 & 0 & -6 \\ 0 & 0 & -43 & -30 & 7 & 0 & -3 & -3 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -7 & 5 & 12 & -14 \\ 0 & 1 & 0 & 0 & 3 & -2 & -5 & -8 \\ 0 & 0 & 1 & 0 & 41 & -30 & -69 & 111 \\ 0 & 0 & 0 & 1 & -59 & 43 & 99 & 159 \end{array} \right)$$

$$A^+ = \begin{pmatrix} -1 & 5 & 12 & -19 \\ 3 & -2 & -5 & 8 \\ 41 & -30 & -69 & 111 \\ -59 & 43 & 99 & -159 \end{pmatrix}$$

$$7 \quad A = \left( \begin{array}{cc|cc} 1 & 3 & -5 & 7 \\ 0 & 1 & 2 & -3 \\ \hline 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array} \right) A^{-1} = \left( \begin{array}{cc} \left( \begin{array}{cc} 1 & 3 \\ 0 & 1 \end{array} \right)^{-1} & -\left( \begin{array}{cc} 1 & 3 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} -5 & 7 \\ 2 & -3 \end{array} \right) \left( \begin{array}{cc} 1 & -2 \\ 0 & 1 \end{array} \right) \\ 0 & \left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right)^{-1} \end{array} \right)$$

$$= \begin{pmatrix} 1 & -3 & 11 & -38 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$8 \quad A = \left( \begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ \hline 5 & 7 & 1 & 8 \\ -1 & -3 & -1 & -6 \end{array} \right) \Rightarrow A^{-1} = \left( \begin{array}{cc} \left( \begin{array}{cc} 2 & 1 \\ 3 & 2 \end{array} \right)^{-1} & 0 \\ -\left( \begin{array}{cc} 1 & 8 \\ -1 & -6 \end{array} \right)^{-1} \left( \begin{array}{cc} 5 & 7 \\ -1 & -3 \end{array} \right) \left( \begin{array}{cc} 2 & 1 \\ 3 & 2 \end{array} \right)^{-1} & \left( \begin{array}{cc} 1 & 8 \\ -1 & -6 \end{array} \right)^{-1} \end{array} \right)$$

$$= \left( \begin{array}{cc} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} & 0 \\ \left( \begin{array}{cc} -5 & 7 \\ 2 & -2 \end{array} \right) & \frac{1}{2} \left( \begin{array}{cc} -6 & -8 \\ 1 & 1 \end{array} \right) \end{array} \right) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ -5 & -7 & -3 & -4 \\ 2 & -2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

⑨

$$\left( \begin{array}{cccc|cccc} 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 4 & 0 & 1 & 0 & 0 \\ 2 & 7 & 6 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & -1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 2 & 2 & -1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 & 1 & -2 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{cccc|cccc} 1 & 2 & 2 & -1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & -1 & 1 & -2 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 3 & 0 & 5 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & -1 & 1 & -2 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & -5/2 & -1/2 & 1/2 & 0 \\ 0 & 3 & 0 & 0 & -7/2 & -3/2 & 5/2 & -5 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 & -1/2 & 1 \\ 0 & 0 & 0 & 1 & 1/2 & 1/2 & -1/2 & 1 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1/6 & 1/2 & -7/6 & 10/3 \\ 0 & 1 & 0 & 0 & -7/6 & -1/2 & 5/6 & 5/3 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 & -3/2 & 1 \\ 0 & 0 & 0 & 1 & 1/2 & 1/2 & -1/2 & 1 \end{array} \right)$$

$$\therefore A^{-1} = \frac{1}{6} \begin{pmatrix} -1 & 3 & -7 & 20 \\ -7 & -3 & 5 & -10 \\ 9 & 3 & -3 & 6 \\ 3 & 3 & -3 & 6 \end{pmatrix}$$

⑩ 求  $A^{-1}$   $A =$

$$\frac{1}{2} \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

方法 1: 令  $B =$

$$\therefore B^{-1} = (E - C)^{-1} = E + C + C^2 + C^3 + C^4$$

$$A^{-1} = (2B)^{-1} = \frac{1}{2} B^{-1} = \frac{1}{2} C + \frac{1}{2} C^2 + \frac{1}{2} C^3 + \frac{1}{2} C^4 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & \frac{1}{32} \\ & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} \\ & & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ & & & \frac{1}{2} & -\frac{1}{4} \\ & & & & \frac{1}{2} \end{pmatrix}$$

$$\text{方法 2: } A = \left( \begin{array}{cc|ccc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ \hline & & 2 & 1 & 0 \\ & & 0 & 2 & 1 \\ & & 0 & 0 & 2 \end{array} \right) = \begin{pmatrix} B & D \\ O & C \end{pmatrix}$$

$$A^0 = \begin{pmatrix} B^{-1} & -B^{-1}OC^{-1} \\ O & C^{-1} \end{pmatrix} = \left( \begin{array}{cc|ccc} \frac{1}{2} & -\frac{1}{4} & -B^{-1} & D & C \\ 0 & \frac{1}{2} & & & \\ \hline & & \frac{1}{2} & -\frac{1}{4} & 8 \\ & & & \frac{1}{2} & -\frac{1}{4} \\ 0 & & & & \end{array} \right)$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & \frac{1}{32} \\ 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

P201. T21

设  $A_{k \times k}$ ,  $C_{rr}$  则  $X = \begin{pmatrix} 0 & A \\ C & 0 \end{pmatrix}$  令  $Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$  才能乘

$$XY \text{ 与 } YX, \quad \text{而 } XY = \begin{pmatrix} AY_3 & AY_4 \\ CY_1 & CY_2 \end{pmatrix} \quad YX = \begin{pmatrix} Y_2C & Y_1A \\ Y_4C & Y_3A \end{pmatrix}$$

$$\text{若 } Y = X^{-1}, \text{ 则 } XY = YX = E \Rightarrow \begin{aligned} CY_1 = 0, Y_1A = 0 &\Rightarrow Y_1 = 0 \\ AY_4 = 0, Y_4C = 0 &\Rightarrow Y_4 = 0 \end{aligned}$$

$$\begin{aligned} AY_3 = Y_3A = E_k \quad Y_3 = A^{-1} \\ \therefore CY_2 = Y_2C = E_r \quad Y_2 = C^{-1} \end{aligned} \therefore X^{-1} = \begin{pmatrix} 0 & C^{-1} \\ A^{-1} & 0 \end{pmatrix}$$

P201. T22

$$\text{将 } X = \begin{pmatrix} 0 & A \\ a_n & 0 \end{pmatrix} A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{n-1} \end{pmatrix} \text{ 由 21 题, (见上面)}$$

$$X^{-1} = \begin{pmatrix} 0 & a^{n-1} \\ A^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_n^{-1} \\ a_1^{-1} & 0 & & & 0 \\ 0 & a_2^{-1} & & & 0 \\ & & \cdots & & \\ 0 & & 0 & a_{n-1}^{-1} & 0 \end{pmatrix}$$

P202. T23.

$$\textcircled{1} \therefore \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 4 & -6 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 & -6 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -23 \\ 0 & 8 \end{pmatrix}$$

②解

$$\left( \begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 & 1 & 1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 & 2 & 0 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & -2 & \frac{1}{2} & -\frac{3}{2} & 1 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & \frac{2}{2} & \frac{2}{3} & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{6} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 1 & 0 \end{array} \right)$$

$$x = A^{-1}B = \begin{pmatrix} 11/6 & 1/2 & 1 \\ -1/6 & -1/2 & 0 \\ 2/3 & 1 & 0 \end{pmatrix}$$

∴

$$(A, B) = \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & \cdots & 1 & 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 1 & 1 & 0 & 1 & 2 & \cdots & 0 & 0 \\ & & & & & \cdots & \cdots & & & & & \\ 0 & 0 & \cdots & & 0 & 1 & 0 & 0 & \cdots & & 1 & 2 \end{array} \right)$$

③ ∵ AX=B, 则 X=A<sup>-1</sup>B, 故

$$X = A^{-1}B = \begin{pmatrix} 1 & -1 & -1 & & 0 \\ 1 & 1 & -1 & \ddots & \\ & \ddots & \ddots & \ddots & -1 \\ & & \ddots & 1 & -1 \\ & 0 & & 1 & 2 \end{pmatrix}$$

所以

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}, \quad \begin{matrix} A_{11}=2 & A_{21}=1 & A_{31}=4 \\ A_{12}=2 & A_{22}=1 & A_{32}=-2 \\ A_{13}=-2 & A_{23}=2 & A_{33}=2 \end{matrix} \\
4 \quad |A| &= 6. \\
X &= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \cdot \frac{1}{6} \begin{pmatrix} 2 & 1 & 4 \\ 2 & 1 & -2 \\ -2 & -2 & 2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -2 & 2 & 8 \\ 4 & 2 & 2 \\ 4 & 5 & 8 \end{pmatrix} \\
\therefore &
\end{aligned}$$

P202. T24

$$\textcircled{1} \because AA^{-1} = A^{-1}A = E \therefore A'(A^{-1})' = (A^{-1})'A' = E \therefore (A')^{-1} = (A^{-1})'$$

若  $A' = A$ , 则  $(A^{-1})' = (A')^{-1} = A^{-1}$ , 即  $A^{-1}$  对称

若  $A' = -A$ , 则  $(A^{-1})' = (A')^{-1} = (-A)^{-1} = -(A^{-1})$  即  $A^{-1}$  反对称

$\textcircled{2}$  若  $A' = -A$ , 那么, 由  $|kA| = k^n|A|$

$$\therefore |A| = |A'| = |A| = (-1)^n |A| = -|A| \therefore |A| = 0$$

于是  $A$  不可逆。

P202. T25

$\textcircled{1}$  若  $A, B$  为上三角形, 则  $A = (a_{ij}), B = (b_{ij})$ , 当  $i > j$  时,  $a_{ij} = 0, b_{ij} = 0$

$$\therefore \text{当 } i > j \text{ 时, } c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} = \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=i}^n a_{ik} b_{kj} = \sum_{k=1}^{i-1} 0 \cdot b_{kj} + \sum_{k=i}^n a_{ik} 0 = 0$$

$\therefore C=AB$  为上三角

若  $A, B$  为下三角形, 则  $A = (a_{ij}), B = (b_{ij})$ , 当  $i < j$  时,  $a_{ij} = 0, b_{ij} = 0$ .  $C = AB, C = (c_{ij})_{n \times n}$

$$\text{当 } i < j \text{ 时, } c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} = \sum_{k=1}^i a_{ik} b_{kj} + \sum_{k=i+1}^n a_{ik} b_{kj} = \sum_{k=1}^i a_{ik} 0 + \sum_{k=i+1}^n 0 b_{kj} = 0 + 0 = 0$$

$\therefore$

$\therefore C=AB$  为下三角

$\textcircled{2} (i)$  若  $A$  为上三角, 考虑  $|A|$  中  $A_{ij}, (i < j)$

$\therefore$

$$A_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1i-1} & a_{1i} & \cdots & a_{1j-1} & a_{ij+1} & \cdots & a_{1n} \\ & \ddots & a_{i-1,i-1} & a_{i-1,i} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ & & 0 & & a_{i+1,i+1} & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ & & & 0 & \cdots & \cdots & & & \\ & & & & \ddots & a_{ij-1} & a_{i+1,j+1} & \cdots & a_{j-1,n} \\ & & & & & 0 & a_{ji+1} & \cdots & a_{jn} \\ & & & & & & a_{j+1,j+1} & \cdots & a_{j+1,n} \\ & & & & & & \cdots & & \\ & & & & & & \cdots & & \\ & & & & & & & & a_{nn} \end{vmatrix} = 0$$

而  $i < j$ ,  $A_{ij}$  位于  $A^*$  的对角线下方,  $\therefore A^*$  上三角, 故  $A^{-1}$  上三角

$\because$  当  $A$  为下三角时,  $A^T$  上三角  $\therefore (A^T)^{-1}$  为上三角, 即  $(A^{-1})^T$  为上三角, 故  $A^{-1}$  为下三角。

P202. T26

$$\therefore AA^* = A^*A = |A|E. \therefore |A^*||A| = |A|^n$$

$$\text{若 } |A| \neq 0, \therefore |A^*| = |A|^{n-1}$$

$$\text{若 } |A| = 0, \text{ 由 P205.18 题 (P4.51.3.2). } AA^* = 0 \therefore \text{秩}(A) + \text{秩}(A^*) \leq 1$$

$$(i) \text{ 若秩}(A) = 0 \Rightarrow A = 0 \Rightarrow A^* = 0 \Rightarrow |A^*| = 0 \therefore |A^*| = |A|^{n-1} (\because n \geq 2)$$

$$(ii) \text{ 若秩}(A) \neq 0 \Rightarrow \text{秩}(A^*) < n \Rightarrow |A^*| = 0 \Rightarrow |A^*| = |A|^{n-1} = 0$$

总之, 各种情形均有  $|A^*| = |A|^{n-1}$

P202. T28

$$\textcircled{1} (AE) = \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -2 & -1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 0 & -1 & 0 & 0 & 1 \end{array} \right)$$



$$\left( \begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right)$$

$$\rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{array} \right) \quad \therefore A^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\textcircled{2} A = \begin{pmatrix} B & B \\ B & -B \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad B^{-1} = -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} B$$

$$\begin{pmatrix} E & O \\ -E & E \end{pmatrix} \begin{pmatrix} B & B \\ B & -B \end{pmatrix} = \begin{pmatrix} B & B \\ O & -2B \end{pmatrix} \quad \text{而} \quad \begin{pmatrix} B & B \\ O & -2B \end{pmatrix}^{-1} = \begin{pmatrix} B^{-1} & \frac{1}{2} B^{-1} \\ 0 & (-2B)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} B & \frac{1}{4} B \\ 0 & -\frac{1}{4} B \end{pmatrix} \begin{pmatrix} E & 0 \\ -E & E \end{pmatrix} = \begin{pmatrix} \frac{1}{4} B & \frac{1}{4} B \\ \frac{1}{4} B & -\frac{1}{4} B \end{pmatrix} = \frac{1}{4} A$$

$$\text{方法③: } \because A^2 = 4A \quad \therefore A^{-1} = \frac{1}{4} A$$

P203. T29

$$\begin{pmatrix} E_m & 0 \\ -A & E \end{pmatrix} \begin{pmatrix} E_m & B \\ A & E_n \end{pmatrix} = \begin{pmatrix} E_m & B \\ 0 & E_n - AB \end{pmatrix}, \text{即} \begin{pmatrix} E_m & 0 \\ A & E_n \end{pmatrix} \begin{pmatrix} E_m & 0 \\ -A & E_n \end{pmatrix} = \begin{pmatrix} E_m - BA & B \\ 0 & P \end{pmatrix}$$

$$\therefore \begin{vmatrix} E_m & B \\ A & E_n \end{vmatrix} = |E_m| |E_n - AB| = |E_n - AB| = |E_m - BA| |E_n| = |E_m - BA|$$

$$\begin{pmatrix} E_m & B \\ A & \lambda E_n \end{pmatrix} \text{则} \begin{pmatrix} \lambda E_m & 0 \\ -A & E_n \end{pmatrix} \begin{pmatrix} E_m & B \\ A & \lambda E_n \end{pmatrix} = \begin{pmatrix} \lambda E_n & \lambda B \\ 0 & \lambda E_n - \lambda B \end{pmatrix}$$

P203. T30

$$\text{又} \begin{pmatrix} E_m & B \\ A & \lambda E_n \end{pmatrix} \begin{pmatrix} \lambda E_m & 0 \\ -A & E_n \end{pmatrix} = \begin{pmatrix} \lambda E_m - \lambda B & B \\ 0 & \lambda E_n \end{pmatrix}$$

$$\therefore \lambda_m |\lambda E_m - AB| = |\lambda E_m| |\lambda E_n - AB| = \begin{vmatrix} \lambda E_m & \lambda B \\ 0 & \lambda E_m - \lambda B \end{vmatrix} = \begin{vmatrix} \lambda E_m & 0 \\ -A & E_n \end{vmatrix} \begin{vmatrix} E_m & B \\ A & \lambda E_n \end{vmatrix}$$

$$= \begin{vmatrix} \lambda E_m - BA & B \\ 0 & \lambda E_n \end{vmatrix} = |\lambda E_m - BA| |\lambda E_n| = |\lambda E_m - BA|$$

$$\therefore |\lambda E_m - AB| = \lambda^{n-m} |\lambda E_m - BA|$$

## 第五章 二次型习题解答

P232. T1

(I) ②) 化标准形,  $f = x_1^2 + 2x_1x_2 + 2x_2^2 + 4x_2x_3 + 4x_3^2$

$$\text{解: } f = (x_1 + x_2)^2 + x_2^2 + 4x_2x_3 + 4x_3^2 \\ = (x_1 + x_2)^2 + (x_2 + 2x_3)^2 + 0$$

$$\text{令} \begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_2 + 2x_3 \\ y_3 = x_3 \end{cases} \quad \text{即} \begin{cases} x_1 = y_1 - y_2 + 2y_3 \\ x_2 = y_2 - 2y_3 \\ x_3 = y_3 \end{cases}$$

$$\text{则 } f = y_1^2 + y_2^2$$

$$\text{用矩阵验算} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}' \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(I) ③ 化标准形  $f = x_1^2 - 3x_2^2 - 2x_1x_2 + 2x_1x_3 - 6x_2x_3$

$$\text{解: } f = (x_1 - x_2 + x_3)^2 - (x_2 - x_3)^2 - 3x_2^2 - 6x_2x_3 \\ = (x_1 - x_2 + x_3)^2 - 4x_2^2 - 4x_2x_3 - x_3^2 \\ = (x_1 - x_2 + x_3)^2 - (2x_2 + x_3)^2$$

$$\text{令} \begin{cases} y_1 = x_1 - x_2 + x_3 \\ y_2 = 2x_2 + x_3 \\ y_3 = x_3 \end{cases} \quad \text{即} \begin{cases} x_1 = y_1 + \frac{1}{2}y_2 - \frac{3}{2}y_3 \\ x_2 = \frac{1}{2}y_2 - \frac{1}{2}y_3 \\ x_3 = y_3 \end{cases}$$

$$\text{则 } f = y_1^2 - y_2^2$$

$$\begin{pmatrix} 1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -1 & -3 & -3 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

验算有:

(I) ④化标准形  $f=8x_1x_4+2x_3x_4+2x_2x_3+8x_2x_4$

$$\begin{cases} x_1 = y_1 + y_4 \\ x_2 = y_2 \\ x_3 = y_3 \\ x_4 = y_1 - y_4 \end{cases} \quad \text{则 } X=C_1YQ$$

$$\begin{aligned} f &= 8(y_1^2 - y_4^2) + 2y_3(y_1 - y_4) + 2y_2y_3 + 8y_2(y_1 - y_4) \\ &= 8y_1^2 - 8y_4^2 + 8y_1y_2 + 2y_1y_3 + 2y_2y_3 - 8y_2y_4 - 2y_3y_4 \end{aligned}$$

$$\begin{aligned} \therefore f &= 8(y_1 + \frac{1}{2}y_2 + \frac{1}{8}y_3)^2 - 8(\frac{1}{2}y_2 + \frac{1}{8}y_3)^2 - 8y_4^2 + 2y_2y_3 - 8y_2y_4 - 2y_3y_4 \\ &= 8(y_1 + \frac{1}{2}y_2 + \frac{1}{8}y_3)^2 - 2(y_2 - \frac{1}{4}y_3 + 2y_4)^2 + 2(-\frac{1}{4}y_3 + 2y_4)^2 - \frac{1}{8}y_3^2 - 8y_4^2 - 2y_3y_4 \\ &= 8(y_1 + \frac{1}{2}y_2 + \frac{1}{8}y_3)^2 - 2(y_2 - \frac{1}{4}y_3 + 2y_4)^2 - 4y_3y_4 \\ &= 8(y_1 + \frac{1}{2}y_2 + \frac{1}{8}y_3)^2 - 2(y_2 - \frac{1}{4}y_3 + 2y_4)^2 + (y_3 - y_4)^2 - (y_3 + y_4)^2 \end{aligned}$$

令

$$\begin{cases} z_1 = y_1 + \frac{1}{2}y_2 + \frac{1}{8}y_3 \\ z_2 = y_2 - \frac{1}{4}y_3 + 2y_4 \\ z_3 = y_3 - y_4 \\ z_4 = y_3 + y_4 \end{cases} \quad \text{即} \quad \begin{cases} y_1 = z_1 - \frac{1}{2}z_2 - \frac{5}{8}z_3 + \frac{3}{8}z_4 \\ y_2 = z_2 + \frac{9}{8}z_3 - \frac{7}{8}z_4 \\ y_3 = \frac{1}{2}z_3 + \frac{1}{2}z_4 \\ y_4 = -\frac{1}{2}z_3 + \frac{1}{2}z_4 \end{cases}$$

$$\text{则 } f = 8z_1^2 - 2z_2^2 + z_3^2 - z_4^2$$

矩阵验算略

(I) ⑤化标准形  $f=x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4$

$$A = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

解:

$$\begin{aligned}
& \left( \begin{array}{c} A \\ E \end{array} \right) \xrightarrow{P_i(2)} \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 2 & 4 & 4 \\ 2 & 0 & 2 & 2 \\ 4 & 2 & 0 & 2 \\ 4 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & -2 & -4 \\ 2 & -1 & -2 & -2 \\ 2 & 1 & -2 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \\
& \rightarrow \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -3 \\ 2 & -1 & -2 & -1 \\ 2 & 1 & -2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \therefore X = \begin{pmatrix} 2 & -1 & -2 & -1 \\ 2 & 1 & -2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix} y
\end{aligned}$$

则  $f = 4y_1^2 - y_2^2 - 4y_3^2 - 3y_4^2$

(I) ⑦化标准形  $f = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_4$

解:  $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$  则  $\left( \begin{array}{c} A \\ E \end{array} \right) \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{aligned}
& \xrightarrow{P(3,(-1))} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}
\end{aligned}$$

$$\text{即令 } X = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix} Y$$

$$\text{则 } f = y_1^2 + 2y_2^2 - 2y_3^2 + y_4^2$$

P233,T2

设秩  $(A) = r$ , 则存在  $C$  满秩

$$C'AC = D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_r \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} = \sum_{i=1}^r d_i E_{ii}$$

那么,  $d_1 E_{11}, d_2 E_{22}, \dots, d_r E_{rr}$  的秩都等于 1, 且为对称的。

$$\therefore A = (C')^{-1} \left( \sum_{i=1}^r d_i E_{ii} \right) C^{-1}$$

$$A = (C')^{-1} \left( \sum_{i=1}^r d_i E_{ii} \right) C^{-1}$$

$$= (C^{-1})' \left( \sum_{i=1}^r d_i E_{ii} \right) C^{-1}$$

$$= \sum_{i=1}^r (C^{-1})' (d_i E_{ii}) C^{-1} = \sum_{i=1}^r B_i$$

$$\text{其中 } B_i = (C^{-1})' (d_i E_{ii}) C^{-1}$$

$$\text{秩}(B_i) = \text{秩}(d_i E_{ii}) = 1, \quad B_i' = (C^{-1})' (d_i E_{ii})' (C^{-1})' = B_i$$

$\therefore A$  为  $r$  个秩为 1 的, 对称阵之和。

P233. T3

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad B = \begin{pmatrix} \lambda_{i1} & & & \\ & \lambda_{i2} & & \\ & & \ddots & \\ & & & \lambda_{in} \end{pmatrix}$$

证明一: 设

$$\text{则 } A = \sum_{i=1}^n \lambda_i E_{ii} \quad B = \sum_{j=1}^n \lambda_{ij} E_{ij}$$

$$\text{又 } i_1, i_2, \dots, i_n \text{ 为 } 1, 2, \dots, n \text{ 的一个排列, 所 } A = \sum_{j=1}^n \lambda_{i_j} E_{i_j i_j}$$

考虑标准单位向量  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$  作  $C = (\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{in})$  则  $C$  的  $n$  列线性无关,  $C$  可逆, 且

$$\begin{aligned}
C &= \sum_{j=1}^n E_{i_j j}, C' = \sum_{j=1}^n E_{j i_j} \\
C'AC &= \left( \sum_{j=1}^n E_{j i_j} \right) \left( \sum_{k=1}^n \lambda_{i_k} E_{i_k i_k} \right) \left( \sum_{l=1}^n E_{i_l l} \right) \\
&= \left( \sum_{j=1}^n \lambda_{i_j} E_{j i_j} \right) \left( \sum_{l=1}^n E_{i_l l} \right) \\
&= \sum_{j=1}^n \lambda_{i_j} E_{j j}
\end{aligned}$$

故 A 与 B 合同

△证法二（归纳法）  $n=1$ , 显然, 设  $n-1$  时命题成立。

考虑  $n$  情形, 设  $i_k = n$

1. 若  $k=n$ , 则  $i_1, \dots, i_{n-1}$  为  $1, 2, \dots, n-1$  的一个排列, 所以

$$\begin{aligned}
\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} &\cong \begin{pmatrix} \lambda_{i_1} & & \\ & \ddots & \\ & & \lambda_{i_1} \end{pmatrix} \\
p'(i_k, n) \begin{pmatrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_{i+1} \end{pmatrix} P(i_k, n) &= \begin{pmatrix} \lambda'_{i-1} & & \\ & \ddots & \\ & & \lambda'_{i_{n-1}} \\ & & & \lambda'_{ik} \end{pmatrix} = B_1
\end{aligned}$$

2. 若  $k < n$ ,

而  $i_1, \dots, i_{n-1}$  为  $1, 2, \dots, n-1$  的一个排列, 所以

$$\begin{aligned}
C'_1 \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n-1} \end{pmatrix} C_1 &= \begin{pmatrix} \lambda'_{i_1} & & \\ & \ddots & \\ & & \lambda'_{i_{n-1}} \end{pmatrix} \\
\therefore \begin{pmatrix} C'_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n-1} \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} \lambda'_i & & \\ & \ddots & \\ & & \lambda'_{i_{n-1}} \\ & & & \lambda_n \end{pmatrix} = B_1 (\because \lambda_n = \lambda_{ik})
\end{aligned}$$

$$\therefore A \cong B_1, B \cong B_1, \therefore A \cong B$$

由归纳原理, 证明完毕。

$$P_{233} 4(1) \Rightarrow \text{若 } A' = A, \text{ 要 } X : a = X'AX = X'AX'' = X'(-A)X = -a$$

$$\therefore a = 0, \text{ 即 } X, X'AX = 0$$

$$4(2) \text{ 令 } X = \varepsilon_i, \text{ 则 } f(\varepsilon_i) = 0 = \varepsilon_i' A \varepsilon_i = a_{ii}, \text{ 即 } a_{ii} = 0$$

$$\text{又令 } X = \varepsilon_i + \varepsilon_j, \text{ 则 } f(\varepsilon_i + \varepsilon_j) = a_{ii} + a_{ij} + a_{ji} + a_{jj} = a_{ij} + a_{ji} = 0$$

$$\therefore a_{ij} = -a_{ji}, \text{ 故 } A' = -A, \text{ 证毕.}$$

$$\text{若 } A' = A, \text{ 则 } a_{ij} = a_{ji}$$

$$\therefore \text{作 } f(x) = X'AX, f(\varepsilon_i) = 0 \Rightarrow a_{ii} = 0 (i = 1, 2, \dots, n)$$

$$\text{又 } f(\varepsilon_i + \varepsilon_j) = 0 = a_{ii} + a_{ij} + a_{ji} + a_{jj} = 2a_{ij}, \text{ 即 } a_{jj} = 0$$

$$4(2) \therefore A = 0$$

P233. T5

设实对称矩阵A,B秩为 $r_A, r_B$ ,正惯性指数为 $P_A, P_B$

$$\therefore A \simeq B \Leftrightarrow r_A = r_B \text{ 且 } p_A = p_B$$

$\therefore 0 \leq p \leq r$ , 当 $r = 0$ 时 只有 $p = 0$ 此1类

当 $r = 1$ 时 有 $p = 0, 1$ , 此2类

当 $r = 2$ 时 有 $p = 0, 2$ , 此3类

当 $r = n$ 时 有 $p = 0, 1, 2, \dots, n$ , 此 $n+1$ 类

$$\text{共有 } 1 + 2 + \dots + (n+1) = C_{n+2}^2 = \frac{1}{2}(n+1)(n+2) \text{ 类}$$

P<sub>233.6</sub> " $\Leftarrow$ "  $f = X'AX$ , ①若 $f$ 的秩=1, 则 $X = C_1 Y$ ,  $C_1$ 可逆. 使

$f = d_1 y_1^2 = (dy_1) \bullet y_1$ , 其中 $dy_1, y_1$ 都是一次齐次多项式  
若

$f$ 的秩=2. 符号差=0. 则 $X = C_2 y$ , ( $C_2$ 可逆) 使

$f = d_1 y_1^2 - d_2 y_2^2, (d_1, d_2 > 0) = (\sqrt{d_1} y_1 + \sqrt{d_2} y_2)(\sqrt{d_1} y_1 - \sqrt{d_2} y_2)$  其中 $\sqrt{d_1} y_1 + \sqrt{d_2} y_2, \sqrt{d_1} y_1 - \sqrt{d_2} y_2$  都是 $x_1, x_2, \dots, x_n$ 的齐次一次式.

" $\Rightarrow$ " 设 $f(x_1, x_2, \dots, x_n) = (a_1 x_1 + a_2 x_2 + \dots + a_n x_n)(b_1 x_1 + b_2 x_2 + \dots + b_n x_n)$

若 $\alpha = (a_1, a_2, \dots, a_n), \beta = (b_1, b_2, \dots, b_n)$ 线性无关, 不妨设 $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0$

$$\text{令 } \begin{cases} y_1 = a_1 x_1 + a_2 x_2 + \dots + a_n x_n \\ y_2 = b_1 x_1 + b_2 x_2 + \dots + b_n x_n \\ \dots\dots\dots \\ y_i = x_i \end{cases} \text{ 则 } f = y_1 y_2$$

$$\text{令 } \begin{cases} y_1 = z_1 + z_2 \\ y_2 = z_1 - z_2 \\ \dots\dots\dots \\ y_i = z_i \end{cases} \text{ 则 } f = z_1^2 - z_2^2, \text{ 秩为2 } \text{ 符号差} = 0$$

$$\text{若 } \alpha, \beta \text{ 线性相关, 不妨设 } \beta = k\alpha \text{ 及 } a_1 \neq 0, \text{ 令 } \begin{cases} y_1 = a_1 x_1 + a_2 x_2 + \dots + a_n x_n \\ y_2 = x_2 \\ \dots\dots\dots \\ y_n = x_n \end{cases}$$

则 $f = k y_1^2$ , 秩为1

$$P_{233} 7(1) A = \begin{pmatrix} 99 & -6 & 24 \\ -6 & 10 & -30 \\ 24 & -30 & 71 \end{pmatrix}$$

$$p_1 = 99 > 0, p_2 = 12834 > 0, p_3 = 20 - 672 - 672 - 288 - 16 - 1960 = -3588 < 0$$

$\therefore A$  正定, 二次型也正定.

$$(2) A = \begin{pmatrix} 10 & 4 & 12 \\ 4 & 2 & -14 \\ 12 & -14 & 1 \end{pmatrix}$$

$$p_1 = 10 > 0, p_2 = 20 - 16 = 4 > 0, p_3 = 20 - 672 - 288 - 16 - 1960 = -3588 < 0$$

$\therefore A$  非正定, 二次型  $X'AX$  非正定

$$(3) \text{ 判定 } f(x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i^2 + \sum_{1 \leq i < j \leq n} x_i x_j \text{ 的正定性}$$

$$A = \frac{1}{2} \begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ 1 & 1 & 2 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 2 \end{pmatrix}$$

解

$$P_k = \frac{1}{2^k} \begin{vmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & 2 \end{vmatrix}_{k \text{ 阶}} = \frac{1}{2^k} (2-1)(2+(K-1)) = \frac{K+1}{2^k} > 0$$

由公式

故  $A$  正定, 二次型  $f(x_1, x_2, \dots, x_n)$  正定

这里顺便发现一个等式

$$\begin{vmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 1 & \dots & 1 & 1 \\ 1 & 1 & 2 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 2 & 1 \\ 1 & 1 & 1 & \dots & 1 & 2 \end{vmatrix}$$

P<sub>233</sub>.7④

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i^2 + \sum_{i=1}^{n-1} x_i x_{i+1}, \text{ 是否正定。}$$

.判别



$$A = \begin{pmatrix} 1 & \frac{1}{2} & \dots & \dots & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & \dots & \dots \\ \dots & \frac{1}{2} & 1 & \dots & \frac{1}{2} \\ 0 & \dots & \dots & \frac{1}{2} & 1 \end{pmatrix}$$

证:

由斜行列式,  $P_k = p_k - 1 - \frac{1}{4}p_k - 2, \therefore P_k - \frac{1}{2}P_{k-1} = \frac{1}{2}(p_{k-1})\frac{1}{2_{n-1}}(p_1 - \frac{1}{2}p_0) = \frac{1}{2_k}$

$$\therefore \frac{1}{2}(P_{k-1} - \frac{1}{2}P_{k-1}) = \frac{1}{2}(\frac{1}{2_{n-1}}) = \frac{1}{2_k}$$

$$\therefore \frac{1}{2_{k-2}}(P_2 - \frac{1}{2}P_1) = \frac{1}{2_k}$$

$$\therefore P_k = \frac{1}{2_{k-1}}P_1 = \frac{k-1}{2_k} \quad \therefore P_k = \frac{k+1}{2^k} > 0, k=1, 2, \dots, n$$

$\therefore A$  正定  $Cf(x_1, x_2, \dots, x_n)$  正定

$$P_{233}8(1)A = \begin{pmatrix} 1 & t & 12 \\ t & 1 & 2 \\ -1 & 3 & 5 \end{pmatrix}$$

$$p_1 = 1 > 0, p_2 = 1 - t^2 > 0, p_3 = 5 - 4t - 1 - 5t^2 = 4t - 5t^2 > 0$$

$$\therefore -1 < t < 1 \text{ 且 } -\frac{4}{5} < t < 0, \text{ 即 } -\frac{4}{5} < t < 0$$

$$\therefore \text{当 } -\frac{4}{5} < t < 0 \text{ 时, } A \text{ 为正定, 相应二次型也正定.}$$

$$P_{233}8 \textcircled{2} x_1^2 + 4x_2^2 + x_3^2 + 2tx_1x_2 + 10x_1x_3 + 6x_2x_3$$

$$A = \begin{pmatrix} 1 & t & 5 \\ t & 4 & 3 \\ 5 & 3 & 1 \end{pmatrix}$$

解:

$$P_1 = 1 > 0, P_2 = 4 - t^2 > 0, \therefore t^2 < 4, \therefore -2 < t < 2$$

$$P_3 = 4 + 30t - 100 - 9 - t^2 > 0, \text{ 即 } t^2 - 30t + 105 < 0 \text{ 又因为 } 15 - 2\sqrt{30} < 15 - 2 \times 6 = 3 > 2$$

$\therefore$  无公共解

0

即对任何  $t$  都有主子式大于 0

$P_{233}$ .9.  $A$  正定  $\Leftrightarrow A$  的主子式全大于 0.

证明:  $\Leftarrow$  此时  $A$  的顺序主子式也大于 0. 所以  $A$  正定 (定理)

$\Rightarrow$  任取的  $i_1, i_2, \dots, i_k$  行,  $i_1, i_2, \dots, i_k$  列作成  $k$  阶主子式

$$B = \begin{pmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \dots & a_{i_1 i_k} \\ a_{i_2 i_1} & a_{i_2 i_2} & \dots & a_{i_2 i_k} \\ \dots & \dots & \dots & \dots \\ a_{i_k i_1} & a_{i_k i_2} & \dots & a_{i_k i_k} \end{pmatrix}, P = |B|$$

设  $f = X'AX$ , 作一个关于  $x_{i_1}, x_{i_2}, \dots, x_{i_k}$  的二次型

$$g(x_{i_1}, x_{i_2}, \dots, x_{i_k}) = f(0, \dots, 0, x_{i_1}, 0, x_{i_2}, 0, x_{i_k}, 0, \dots)$$

$$(x_{i_1}, x_{i_2}, \dots, x_{i_k}) B \begin{pmatrix} x_{i_1} \\ x_{i_2} \\ \dots \\ x_{i_k} \end{pmatrix}$$

$B$  是  $g$  的矩阵, 因为任给  $(x_{i_1}, x_{i_2}, \dots, x_{i_k}) \neq 0$

$$\therefore g(c_{i_1}, c_{i_2}, \dots, c_{i_k}) = f(0, \dots, 0, c_{i_1}, 0, \dots, 0, c_{i_k}, 0, \dots) > 0$$

$\therefore B$  为  $K$  的正定矩阵,  $|B| > 0$

$P_{233, 10}$  证: 设  $A = (a_{ij})_{n \times n}$ , 那么  $tE + A$  的第  $L$  个顺序主子式

$$\tilde{p}_R(t) = \begin{vmatrix} t + a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & t + a_{22} & \dots & a_{2k} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & t + a_{kk} \end{vmatrix}, = t_k + b_{k1}t_k - 1 + \dots b_{kk}$$

$$\lim_{t \rightarrow \infty} \tilde{p}_k(t) = +\infty, \forall M > 0$$

是一个  $t$  的多项式 (函数), 且  $t \rightarrow \infty$

$\therefore \exists N_k$ , 当  $t > N_k$  后, 恒有  $\tilde{p}_k(t) > M > 0$

取  $N_0 = \min\{N_1, N_2, \dots, N_n\}$  则当  $t > N_0$ , 恒有

$$\tilde{p}_1(t) > 0, \tilde{p}_2(t) > 0, \dots, \tilde{p}_n(t) > 0$$

$tE + A$  正定

$P_{233, 11}$ .  $A$  正定, 证明  $A^{-1}$  正定

证:  $\therefore A$  可逆  $CAE$  正定  $C$  存在  $C$  可逆使

$$C'AC = E$$

$$\therefore (C'AC)^{-1} = E^{-1} = E$$

$$C^{-1}A^{-1}(C')^{-1} = E, \text{ 取 } G = (C')^{-1}, \text{ 那么 } G' = ((C')^{-1})' = (C'')^{-1} = C^{-1}$$

即  $\therefore G'A^{-1}G = E$ , 则  $A^{-1} \simeq E$ ,  $\therefore A^{-1}$  正定.

$P_{234.12}$  考虑  $(tE + A)$ , 因为  $t$  充分大后 (10题  $P_{5.77.7.2}$ )  $tE + A > 0$

故可设  $t_0 > 0$ , 且  $|t_0 E + A| > 0$ . 又因为当  $t=0$  时,  $|A| < 0$  所以

$\varepsilon \in (0, t_0)$ ,  $|\varepsilon E + A| = 0$ , 所以有  $X \neq 0$ , 使  $(\varepsilon E + A)X = 0$

即  $X'(\varepsilon E + A)X = 0$ . ( $\because x \neq 0, \therefore x'x > 0, \varepsilon x'x > 0$ ). 得到  $X'AX = -\varepsilon X'X < 0$

$P_{234.13}$  证, 设 有  $f_1 = X'AX, f_2 = X'BX$

则由  $A, B$  正定, 有  $f_1, f_2$  正定, 即要  $x \neq 0, X'AX > 0, X'BX > 0$

作  $f = f_1 + f_2 = X'(A+B)X$

也有任  $X \neq 0, f = X'AX + X'BX > 0$

所以  $f$  正定, 即  $(A+B)$  正定

$P_{234.14} f = X'AX \geq 0 \Leftrightarrow$  秩  $r =$  惯性指数  $P$

证: 设  $X=CY$ , 使  $f = X'AX = y_1^2 + y_2^2 + \dots + y_p^2 + y_{p+1}^2 - \dots - y_r^2$

“充分性  $\Rightarrow$ ” 若  $P=r$ , 则负系数平方项不出现

$$X_0 = \begin{pmatrix} c_1 \\ \dots \\ c_n \end{pmatrix} \neq 0, \text{必有 } y_0 = C^{-1}X_0, f \text{ 在 } X_0 \text{ 的值为}$$

$$\therefore f_1 x = x_0 = X'_0 A X_0 = y_1^2 + \dots + y_r^2 \geq 0$$

任取  $\therefore f$  半正定

$$p < r, \text{取 } y_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \varepsilon_r, x_0 = cy_0 = \begin{pmatrix} c_{1r} \\ c_{2r} \\ \vdots \\ c_{nr} \end{pmatrix} \neq 0$$

“必要性  $\Rightarrow$ ”, 反设

一方面,  $f = X'_0 A X_0 \geq 0$

$$f = X'_0 A X_0 = y_0' \begin{pmatrix} 1 & & & & & & \\ & \dots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & \dots & & \\ & & & & & -1 & \\ & & & & & & 0 \\ & & & & & & & \dots \\ & & & & & & & & 0 \end{pmatrix} y_0 = -1 < 0$$

另一方面,

矛盾!  $\therefore p = r$

$$P_{234.15} \text{ 证明 } f = n \sum_{i=1}^n x_i^2 - \left( \sum_{j=1}^n x_j \right)^2 \geq 0$$

$$f = \sum_{j=1}^n x_j^2 - 2 \sum_{1 \leq i < j \leq n} x_{ij}$$

证法一:因为

$$f = \sum_{1 \leq i < j \leq n} (x_i - x_j)^2$$

恰好有

故任取  $(c_1, c_2, \dots, c_n) \neq 0$ , 必有

$$f(c_1, c_2, \dots, c_n) = \sum (c_i - c_j)^2 \geq 0$$

$\therefore f$  半正定  $B$

$$f = X'AX, \text{ 则 } A = \begin{pmatrix} n-1 & -1 & -1 & -1 \\ -1 & n-1 & \dots & -1 \\ \dots & \dots & \dots & \dots \\ -1 & -1 & \dots & n-1 \end{pmatrix}$$

证法二:设

因此  $A$  的任意  $k$  阶主子式为

$$Q_k = \begin{vmatrix} n-1 & -1 & \dots & -1 \\ 1 & n-1 & \dots & -1 \\ \dots & \dots & \dots & \dots \\ -1 & -1 & \dots & n-1 \end{vmatrix}_{k \text{ 阶}} \Rightarrow (n-1-1)^{k-1} (n-1+(k-1)(-1))$$

恒有  $Q_1, Q_2, \dots, Q_{n-1} > 0, Q_n = 0$

$\therefore A$  的所有主子式大于或等于 0 即  $f = X'AX$  半正定

P234.1 证: 首先  $x_1, x_2$  线性无关,

(反设  $x_1, x_2$  线性相关, 不妨设有  $X_2 = kX_1, k \in R$ )

$$\therefore X_2'AX_2 = (kx_1')A(kx_1) = k_2(x_1'Ax_1) > 0$$

与  $x_2'Ax_2 < 0$  矛盾。

$$x(t) = x_1 + t(x_2 - x_1) = tx_2 + (1-t)x_1 \neq 0, (\text{对任何 } t)$$

$\therefore$  二次型  $f = x'Ax$  在  $x(t)$  的值为

$$q(t) = x'(t)Ax(t) = (tx_2' + (1-t)x_1')A(tx_2 + (1-t)x_1)$$

$$= (t(x_2' - x_1')A(t(x_2 - x_1) + x_1))$$

$$= t_2(x_2 - x_1)'A(x_2 - x_1) - 2tx_1'A(x_2 - x_1) + x_1'Ax_1$$

$$\sigma(0) = x_1'Ax_1 > 0, \sigma(1) = x_2'Ax_2 < 0$$

是  $t$  的初等连续函数,

所有  $t_0$ , 使  $\phi(t_0) = 0$ . 令  $x_0 = x(t_0) = x_1 + t_0(x_2 - x_1) \neq 0$ .

则有  $x_0'Ax_0 = \phi(t_0) = 0$ , 证毕。

P234 补 1①化标准形.  $f = x_1x_{2n} + x_2x_{2n-1} + \dots + x_nx_{n+1}$

$$\begin{array}{l} \text{解} \begin{cases} x_1 = y_1 + y_{2n} \\ x_2 = y_2 + y_{2n-1} \\ \dots\dots\dots \\ x_n = y_n + y_{n+1} \\ x_{n+1} = y_n - y_{n-1} \\ \dots\dots\dots \\ x_{2n-1} = y_2 - y_{2n-1} \\ x_{2n} = y_1 - y_{2n} \end{cases} \end{array} \quad \text{即} \quad X = \begin{pmatrix} 1 & & & & & & 1 \\ & 1 & & & & & \\ & & \dots & & \dots & & \\ & & & 1 & 1 & & \\ & & & 1 & -1 & & \\ & & \dots & & & \dots & \\ & 1 & & & & & -1 \\ 1 & & & & & & -1 \end{pmatrix}$$

$$Y = cy \quad \text{则有}$$

$$f = y_1^2 + y_2^2 + \dots + y_n^2 - y_{n+1}^2 - \dots - y_{2n}^2$$

$$C' \left( \frac{1}{2} \begin{pmatrix} & & & 1 \\ & & \dots & \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix} \right) C = \begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix}$$

验算

$$\text{令} \quad H = \begin{pmatrix} & & & 1 \\ & & \dots & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \text{则} H^2 = En, A = \begin{pmatrix} 0 & \frac{1}{2}H \\ \frac{1}{2}H & 0 \end{pmatrix}, C = \begin{pmatrix} E & H \\ H & -E \end{pmatrix}$$

$$\therefore C'AC = \begin{pmatrix} E' & H' \\ H' & -E' \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2}H \\ \frac{1}{2}H & 0 \end{pmatrix} \begin{pmatrix} E & H \\ H & -E \end{pmatrix} = \begin{pmatrix} E & H \\ H & -E \end{pmatrix} \begin{pmatrix} \frac{1}{2}E & -\frac{1}{2}H \\ \frac{1}{2}H & \frac{1}{2}E \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$$

P234 补 1②化标准形  $f = x_1x_2 + x_2x_3 + \dots + x_{n-1}x_n$

解:若设  $y_1 = \frac{1}{2}(x_1 + x_2 + x_3), y_2 = \frac{1}{2}(x_1 - x_2 + x_3)$ , 则

$$y_1^2 - y_2^2 = x_1x_2 + x_2x_3$$

$$(1) \quad (1) \quad \text{若 } n \text{ 是偶数, 则 } \left\{ \begin{array}{l} y_1 = \frac{1}{2}(x_1 + x_2 + x_3) \\ y_2 = \frac{1}{2}(x_1 - x_2 + x_3) \\ y_3 = \frac{1}{2}(x_3 + x_4 + x_5) \\ y_4 = \frac{1}{2}(x_3 - x_4 + x_5) \\ \dots\dots\dots \\ y_{n-3} = \frac{1}{2}(x_{n-3} + x_{n-2} + x_{n-1}) \\ y_{n-2} = \frac{1}{2}(x_{n-3} - x_{n-2} + x_{n-1}) \\ y_n = \frac{1}{2}(x_{n-1} - x_n) \end{array} \right\}$$

$$c_1 = \frac{1}{2} \left( \begin{array}{cc|cc|cc|cc} 1 & -1 & 1 & & & & & \\ 1 & -1 & 1 & & & & & \\ \hline & & 1 & 1 & 1 & & & \\ & & 1 & -1 & 1 & & & \\ \hline & & & & \dots & & & \\ \hline & & & & & 1 & 1 & \\ & & & & & 1 & -1 & \\ \hline & & & & & & & 1 & 1 \\ & & & & & & & 1 & -1 \end{array} \right)$$

即,  $Y = C_1 X$

显然  $|C_1| = \frac{1}{2^n} (-2)^{\frac{1}{2}} \neq 0$ , 令  $C = C_1^{-1}$

则  $X = CY$  使

$$f = y_1^2 - y_2^2 + y_3^2 - y_4^2 + \dots + y_{n-3}^2 - y_{n-2}^2 + y_{n-1}^2 - y_n^2$$

$\Delta(ii)$  若  $n$  为奇数, 同理

补 P<sub>234.1</sub>③)(也可直接证明, 或归纳证明)

P234 补 1.④ 
$$f = \sum_{i=1}^n (x_i - \bar{X})^2, \bar{X} = \frac{1}{n} \sum_{i=1}^n x_i$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_{n-1} \\ y_n \end{pmatrix} = y = \frac{1}{n} \begin{pmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & \dots & n-1 & -1 \\ 0 & 0 & \dots & \dots & 1 \end{pmatrix} X = \frac{1}{n} C_3 X \text{ 或 } X = \left(\frac{1}{n} c_3\right)^{-1} y$$

$$= \begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 2 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} y$$

令

$$\because y_i = x_i - \frac{1}{n} \sum_{i=1}^n x_i = x_i - \bar{x}, y_n = x_n, \sum y_i = \sum x_i - (n-1)\bar{x} = \bar{x}$$

$$\therefore f = \sum_{i=1}^{n-1} y_i^2 + (x_n - \bar{x})^2 = \sum_{i=1}^{n-1} y_i^2 + (y_n - \sum_{i=1}^n y_i)^2 = 2 \left( \sum_{i=1}^{n-1} y_i^2 + \sum_{1 \leq i < j \leq n-1} y_i y_j \right)$$

参照P<sub>234</sub>.1③(5.75.5.3)令

$$Z = C_4 Y = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{3} & \dots & \frac{1}{3} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \frac{1}{n-1} & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix} Y$$

$$\text{则 } f = 2z_1^2 + \frac{3}{2}z_2^2 + \frac{4}{3}z_3^2 + \dots + \frac{n}{n+1}z_{n-1}^2$$

$$x = \left(\frac{1}{n} c_3\right)^{-1} y = \left(\frac{1}{n} c_3\right)^{-1} c_4^{-1} y = n(c_4 c_3)^{-1} y$$

其中

矩阵验算略.

$$\text{P}_{234} \text{补 2, 不妨设 } \text{秩}(A) = r, \text{ 且 } \begin{vmatrix} a_{11} & \dots & a_{1r} \\ \dots & \dots & \dots \\ a_{r1} & \dots & a_{rr} \end{vmatrix} \neq 0 (\text{左上角 } r \text{ 阶子式})$$

$$\therefore \text{作 } X = C_1 Y, \text{ 其中 } C_1 = \begin{pmatrix} a_{11} & \dots & a_{1r} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{r1} & \dots & a_{rr} & \dots & a_{rn} \\ 0 & \dots & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

$$\text{那么, } f = y_1^2 + y_2^2 + \dots y_r^2 + \delta_{r+1}^2 + \dots + \delta_s^2 (\delta_i \text{ 为 } y_1, \dots y_r \text{ 的一次式})$$

作一个  $f_1(y_1, y_2, \dots, y_r) = f$  被为  $r$  元二次型  $\diamond$  那么  $\diamond$  任取  $\diamond c_1, c_2, \dots, c_r \neq 0$   
 那么, 必有  $f_1(c_1, c_2, \dots, c_r) = f(c_1, \dots, c_r, \dots, 0) > 0, \therefore f_1$  是一个  $r$  元正定二次型

$$\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_r \end{pmatrix} = G \begin{pmatrix} z_1 \\ z_2 \\ \dots \\ z_r \end{pmatrix} \text{ 使 } f_1(y_1, y_2, \dots, y_r) = c_1^2 + c_2^2 + \dots + z_r^2$$

$$\text{令 } \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix} = \begin{pmatrix} G & o \\ o & E_{n-r} \end{pmatrix} \begin{pmatrix} z_1 \\ \dots \\ z_r \\ \dots \\ z_n \end{pmatrix}, C_2 = \begin{pmatrix} G & o \\ o & E_{n-r} \end{pmatrix} \text{ 可逆}$$

$$\text{则 } f = y_1^2 + \dots + y_r^2 + \delta_{r+1}^2 + \dots + \delta_s^2$$

$$\text{且 } X = C_1 Y = C_1 C_2 Z = CZ (C = C_1 C_2 \text{ 可逆 } \diamond)$$

必有  $\therefore f$  的正惯性指数  $= r = \text{秩}(A)$

$$P_{234} \text{ 补 3 (先讲补 2), } f = l_1^2 + \dots + l_p^2 - l_{p+1}^2 - \dots - l_{p+q}^2$$

$$\text{证: 设 } l_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$$

$$\text{并设 } X = CY, (C \text{ 可逆}) \text{ 使 } f = y_1^2 + y_2^2 + \dots + y_{p'}^2 - y_{p'+1}^2 - \dots - y_{p'+q}^2,$$

$$p' > p \text{ 作线性方程组 } \left\{ \begin{array}{l} l_1 = 0 \\ \dots \\ l_p = 0 \\ \dots \\ y_{p'+1} = 0 \\ \dots \\ y_n = 0 \end{array} \right\}$$

那么 反设

$$y_i = b_{i1}x_1 + \dots + b_{in}x_n$$

$$Y = C^{-1}X$$

共有  $p + (n - p') = n - (p' - p) < n$  的方程

存在非零时,

$$X_0 = \begin{pmatrix} c_1 \\ \dots \\ c_n \end{pmatrix} \neq 0, y_0 = c^{-1}X_0 \neq 0, \therefore y_{p'+1} = \dots = y_n = 0 \therefore y_0 \text{ 的前 } p' \text{ 个分量不全为 } 0$$

$$\therefore \text{一方面 } f = -l_{p+1}^2 - \dots - l_{p+q}^2 \leq 0$$

$$\text{另一方面 } f = y_1^2 + \dots + y_{p'}^2 > 0 (i \text{ 不全为 } 0)$$

矛盾, 所有  $p' \leq p$



同理,负惯性指数  $q' \leq q$

另推论:如本例形式二次型,例  $p+q \geq r$ (秩)

$$\begin{aligned} \text{p}_{235} \text{补4证明: } & \because T'AT = A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, T = \begin{pmatrix} E & X \\ o & E \end{pmatrix} \\ & = \begin{pmatrix} E & X \\ o & E \end{pmatrix} \begin{pmatrix} A_{11} & A_{11}X + A_{12} \\ A_{21} & A_{21} + A_{22} \end{pmatrix} + \begin{pmatrix} E & o \\ X' & E \end{pmatrix} \begin{pmatrix} A_{11} & A_{11}x + A_{12} \\ A_{21} & A_{21}x + A_{22} \end{pmatrix} \\ & = \begin{pmatrix} A_{11} & A_{11}x + A_{12} \\ x'A_{11} + A_{21} & * \end{pmatrix} \end{aligned}$$

$$A_{11}x + A_{12} = 0 \text{ 则 } x = -A_{11}^{-1}A_{12}, (\because A_{12} = A_{21})$$

$$\therefore x'A_{11} + A_{21} = (-A_{12}', A_{11}^{-1})A_{11} + A_{21} = -A_{21}A_{11}^{-1}A_{11} + A_{21} = 0$$

设

$$\text{取 } T = \begin{pmatrix} E & -A_{11}^{-1}A_{12} \\ o & E \end{pmatrix} \text{ 即合要求}$$

P<sub>235</sub>, 补 5, 若 n=1, 显然 A=0

若 n=2, A=0, 显然

$$A \neq o, \text{ 则 } \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ 成立}$$

【】

由于 A<sub>2</sub> 仍为反对称

$$T'_2 A_2 T_2 = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \dots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \\ & & & & & 0 \\ & & & & & & \dots \\ & & & & & & & 0 \end{pmatrix}$$

故归纳假设 A<sub>2</sub>:

$$\therefore \text{取 } C = P_{(s,t)} P_{(t,2)} T \begin{pmatrix} \frac{1}{a} & & & \\ & 1 & & \\ & & \dots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} E_2 & 0 \\ 0 & T_2 \end{pmatrix} \text{ 则}$$

$$C'AC = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \dots & & & \\ & & & 0 & 1 & \\ & & & -1 & 0 & \\ & & & & & 0 \\ & & & & & \dots \\ & & & & & & 0 \end{pmatrix}$$

证毕.

$P_{235}$  补, 6 由习题第 10 题(5.77.7.2), 一定存在  $C_1 > 0$ , 当  $t > c_2$  时  $C_1 E - A$  永为正定

$C > \max\{C_1, C_2\}$  那么同时  $CE + A, CE - A$  正定,

即要  $X_0 \neq 0, x_0'(CE + A)X_0 > 0 \Rightarrow 2CX_0'X_0 < X_0'AX_0$

$X_0'(CE - A)X_0 > 0 \Rightarrow CX_0'X_0 < X_0'AX_0$

即  $-CX_0'X_0 < X_0'AX_0 < CX_0'X_0$

取  $\therefore$  对每个  $X$ , 有  $|X'AX| < CX'X$

$$P_{235} \text{ 补 } 7, 1) \quad T = \begin{pmatrix} 1 & & & * \\ & 1 & & \dots \\ & & 1 & \\ & & & \dots \\ 0 & & & & 1 \end{pmatrix}, B = T'AT$$

$$T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix}, A = \begin{pmatrix} A_1 & A_2 \\ A_2' & A_3 \end{pmatrix} \text{ 其中 } T_1, A_1 \text{ 为 } K \text{ 阶方阵.}$$

将  $T$  分块

则  $B$  的第  $K$  个顺序主子式的矩阵为

$$T'AT = \begin{pmatrix} T_1' & 0 \\ T_3' & T_2' \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_2' & A_3 \end{pmatrix} \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} = \begin{pmatrix} T_1' & 0 \\ T_3' & T_2' \end{pmatrix} \begin{pmatrix} A_1 T_1 & * \\ A_2' T_1 & * \end{pmatrix} = \begin{pmatrix} T_1' A_1 T_1 & * \\ * & * \end{pmatrix}$$

的左上角  $k$  阶方阵, 即为  $T_1' A_1 T_1$

$\therefore B$  的第  $k$  个顺序主子式  $= |T_1' A_1 T_1| = |T_1|^2 |A_1| = |A_1|$ , 为  $A$  的第  $k$  个顺序主子式. 证毕

2) 归纳证明,  $n=1$  显然, 设  $n-1$  成立, 考虑  $n$  情形

设  $A = \begin{pmatrix} A_1 & \alpha \\ \alpha' & a_{\cos} \end{pmatrix}$  由  $A_1$  满足条件, 存在  $T_1$  特殊上三角, 使  $D_1 = T_1' A_1 T_1$  为对角

设  $G = \begin{pmatrix} T_1 & 0 \\ 0 & 1 \end{pmatrix}$  仍为特殊上三角  $\diamond$  使  $G'A'G = \begin{pmatrix} D_1 & T_1'\alpha \\ \alpha'T_1 & a_{nn} \end{pmatrix} = B$

$\because |D_1| = |T_1|^2 |A_1| \neq 0 \therefore D_1$  可逆,

又  $H = \begin{pmatrix} E_{n-1} & -D^{-1} & T_1'\alpha \\ 0 & 1 & \end{pmatrix}$  仍为特殊上三角, 且

$$H'BH = \begin{pmatrix} E_{n-1} & 0 \\ -\alpha'T_1D_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ \alpha'\beta T_1 & a_{nn-x} \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & 6 \end{pmatrix} = D, \text{ 为对角矩阵}$$

故取  $C=GH$  仍为特殊上三角, 且  $C'AC=D$  为对角, 证毕.

3)

$\because A$  的顺序主子式  $P_1, C_2, \dots, P_n$  全大于 0, 故存在特殊上三角  $T$  使  $D = T'AT$  为对角, 于是

$$D \text{ 与 } A \text{ 的顺序主子式值相等, 设 } D = \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & \dots \\ & & & d_n \end{pmatrix}$$

则  $d_1, d_2, \dots, d_k = p_k > 0, k = 1, 2, \dots, n$ , 推出  $d_1, d_2, \dots, d_n > 0$

所以  $D$  正定, 即  $A$  正定, 证毕.

$$f = \begin{vmatrix} A & y \\ y' & 0 \end{vmatrix} = \begin{vmatrix} \begin{pmatrix} A & y \\ y' & 0 \end{pmatrix} \begin{pmatrix} E & -A^{-1}y \\ 0 & 1 \end{pmatrix} \end{vmatrix} = \begin{vmatrix} A & 0 \\ y' & -y'A^{-1}y \end{vmatrix} = (-|A|) \cdot y'A^{-1}y$$

$P_{236}$  补 8, 1),

$\because A$  正定  $i$  已知  $j \cdot CA^{-1}$  正定  $i$  见习题第 11 题  $P5.76.6.4$ ), 故对任一组  $y \neq 0$  值.

$y'A^{-1} > 0, \therefore f(y_1, y_2, \dots, y_n) = (-|A|)y'A^{-1}y < 0, (\because |A| > 0)$

$\therefore f$  是负定二次型

$$A = \begin{pmatrix} A_1 & \alpha \\ \alpha' & a_{nn} \end{pmatrix}, B_1 = \begin{pmatrix} A_1 & \partial \\ \partial' & 0 \end{pmatrix}, B_2 = \begin{pmatrix} A_1 & 0 \\ \alpha' & a_{nn} \end{pmatrix}$$

2) 设

$$\therefore |A| = |B_1| + |B_2| = (-|A_1|)\partial'A_1^{-1}\partial + a_{nn}|A_1| \leq a_{nn}|A_1| = q_{nn}p_{n-1}$$

$\because A_1$  仍然正定  $\therefore |A_1| \leq a_{n-1}p_{n-1}$

如此下去 则  $|A| \leq a_{nn}p_{n-1} \leq a_{nn}q_{n-1}p_{n-2} \leq a_{nn} \dots a_{33}a_{22}p_1 = a_{nn} \dots a_{22}a_{11}$

3) 即  $|A| \leq a_{11}a_{22} \dots a_{nn}$

作  $A = T'T = T'ET$ , 则  $A$  正定 且  $a_{ii} = \sum_{k=1}^n t_{ki}^2$

$$\therefore |A| = |T|^2 \leq \prod_{i=1}^n a_{ii} = \prod_{i=1}^n \sum_{k=1}^n t_{ki}^2 = \prod_{i=1}^n (t_{1i}^2 + t_{2i}^2 + \dots + t_{ni}^2).$$

4)

$P_{236}$  补9 (必要性)

$$A \geq 0 \Rightarrow C \text{ 可逆}, C'AC = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \geq 0 \therefore d_i \geq 0$$

$$\therefore |C|^2 |A| = d_1 d_2 \cdots d_n \geq 0 \therefore |A| \geq 0.$$

$$\text{作 } g(x_{i_1}, \dots, x_{i_2}, \dots, x_{i_k}) = f(0 \dots x_{i_1}, \dots, 0 x_{i_2}, \dots, x_{i_k}, 0)$$

$$\text{则 } g(x_{i_1}, \dots, x_{i_k}) \text{ 半正定, } g \text{ 的矩阵为 } \begin{pmatrix} a_{i_1 i_1} & \cdots & a_{i_1 i_k} \\ \cdots & \cdots & \cdots \\ a_{i_k i_1} & \cdots & a_{i_k i_k} \end{pmatrix} = A_1$$

$$\therefore A_1 \geq 0 \therefore |A_1| \geq 0$$

$$D = \begin{vmatrix} a_{11+\lambda} & a_{12} & \cdots & a_{1n} \\ a_2 1 & a_{22+\lambda} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn+\lambda} \end{vmatrix} = \lambda^n + a_1 \lambda^{n-1} + \dots + a_k \lambda^{n-k} + \dots + a_n \text{ 中 } \lambda^{n-k} \text{ 的系数 } a_k$$

这样取到在  $0$  中主对角线上任取  $n-k$  项中的  $\lambda^{n-k}$  的系数  $a_k$  项所在的行和列, 得一个  $K$  级子式 (含入)  $DK$ , 由于是  $\lambda^{n-k}$  的系数  $a_k$  故  $K$  的子式  $D$  中只能取所有的常数项 (即令

$DK$  中的  $\lambda = 0$ , 这正是  $D$  的一个  $K$  级主子式, 要是  $\lambda^{n-k}$  的系数中的一元, 故  $a_k$  为  $D$  的所有  $k$  阶主子式之和, 如

$$a_1 = a_{11} + a_{22} + \dots + a_{nn} \text{ 等}.$$

现在考虑任意  $\varepsilon > 0$ ,  $A + \varepsilon E$ , 它的  $m$  阶顺序主子式, 为  $A$  的右上角的  $m$  阶方阵,  $A_r$  作:

$$|A_k + \varepsilon E_k| = \varepsilon^k + b_1 \varepsilon^{k-1} + \dots + b_k$$

由于  $\varepsilon^{k-i}$  的系数  $b_i$  是  $A_k$  的一切  $i$  阶主子式之和, 而  $A_k$  的主子式仍为  $A$  的主子式,

$$\text{由充分条件, } b_i \geq 0, \therefore |A_k + \varepsilon E_k| \geq \varepsilon^k > 0$$

因此  $A + \varepsilon E$  正定, 故对任何  $X \neq 0$ ,  $X'(A + \varepsilon E)X > 0$

$$\therefore \lim_{\varepsilon \rightarrow \infty} X'(A + \varepsilon E)X = X'AX \geq 0 \text{ (边续性)}$$

所以  $A$  半正定.

## 第六章 线性空间习题解答

P267.1 设  $M \subseteq N$ , 证明:  $M \cap N = M, M \cup N = N$

$$\text{证: } \forall x \in M \Rightarrow x \in N = M \Rightarrow x \in M \cap N \Rightarrow M \subseteq M \cap N$$

$$\forall x \in M \cap N \Rightarrow x \in M \Rightarrow M \cap N \subseteq M, \therefore M \cap N \subseteq M$$

$$\therefore M \cap N = M$$

$$\text{又} \forall x \in N \Rightarrow x \in M \cup N \Rightarrow N \subset M \cup N$$

$$\forall x \in M \cup N \Rightarrow x \in N \text{ 或 } x \in M \subseteq N \Rightarrow x \in N$$

$$\therefore M \cup N = N$$

$$\text{P267.2 证: ① } M \cap (N \cup L) = (M \cap N) \cup (M \cap L)$$

$$\text{② } M \cup (N \cap L) = (M \cup N) \cap (M \cup L)$$

①:

$$\text{证(1): } x \in \text{左} \Leftrightarrow x \in M \text{ 且 } x \in N \cup L \Leftrightarrow x \in M \text{ 且 } (x \in N \text{ 或 } x \in L) \Leftrightarrow x \in M \cap N \text{ 或 } x \in M \cap L$$

$$\Leftrightarrow x \in \text{右} \Rightarrow \text{反过来。证毕}$$

$$\text{证(2): } x \in \text{左} \Leftrightarrow x \in M \text{ 或 } x \in N \cap L \Leftrightarrow x \in M \text{ 或 } (x \in N \text{ 且 } x \in L)$$

$$\Leftrightarrow x \in M \cup N \text{ 且 } x \in M \cup L \Leftrightarrow x \in \text{右。证毕}$$

P267.3①不做成,因为2个n次多项式相加不一定是n次多项式,如

$$(x'' + x + 1) + (-x'' + x - 2) = 2x - 1$$

$$f(A) + g(A) = h_1(A), (h_1(x) = f(x) + g(x) \text{ 为多项式})$$

$$\text{②做成,因为 } kf(A) = h_2(A), (h_2(x) = kf(x) \text{ 为多项式})$$

做成. 因为实对称(反对称, 上三角, 下三角)之和之倍数仍为实对称

③(反对称, 上三角, 下三角)故做成线性空间

④不做成, 设  $V = \{\alpha \mid \alpha \text{ 为平面上不平行 } \beta \text{ 的向量}\}$

⑤不做成, 违反定义 3.5)  $\because 1\alpha = \alpha$ , 但这里  $1\alpha = 0$ 。取  $\alpha \neq 0$  即得矛盾。

$$(a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2, b_1 + b_2 + a_1 a_2)$$

$$k \circ (a_1, b_1) = (ka_1, kb_1 + \frac{1}{2}k(k-1)a_1^2)$$

P267.3⑤

解: 显然  $V \neq \emptyset$

以及2个封闭的代数运算  $2^0$

验证  $3^0$  先设  $\alpha = (a_1, b_1), \beta = (a_2, b_2), r = (a_3, b_3)$ , 及  $k, t \in R$

$$(1) \alpha \oplus \beta = \beta \oplus \alpha = (a_2 + a_1, b_2 + b_1 + a_2 a_1)$$

$$(2) (\alpha \oplus \beta) + r = ((a_1 + a_2) + a_3, (b_1 + b_2 + a_1 a_2) + b_3 + (a_1 + a_2) a_3)$$

$$\dots = (a_1 + a_2 + a_3, b_1 + (b_2 + b_3 + a_2 a_3))$$

$$\dots \alpha \oplus (\beta \oplus r) = (a_1 + (a_2 + a_3), b_1 + (b_2 + (b_3 + a_2 a_3) + a_1(a_2 + a_3)))$$

$$\dots = (a_1 + a_2 + a_3, b_1 + b_2 + b_3 + a_2 a_3 + a_1 a_2 + a_1 a_3) = (\alpha + \beta) + r$$

$$(3) 0 = (0, 0), \alpha + 0 = (a_1 + 0, b_1 + 0 + a_1 \cdot 0) = (a_1, b_1) = \alpha$$

$$(4) \alpha \text{ 的负为 } -\alpha = (-a_1, a_1^2 - b_1)$$

$$\dots \alpha \oplus (-\alpha) = (a_1 + (-a_1), b_1 + (a_1^2 - b_1) + a_1(-a_1)) = (0, 0) = 0$$

$$(5) 1 \circ \alpha = (1 \circ a_1, 1 \circ b_1 + \frac{1}{2}1 \circ (1-1)a_1^2) = (a_1, b_1) = \alpha$$

$$(6) k \circ (l \circ \alpha) = k \circ (la_1, lb_1 + \frac{1}{2}l(l-1)a_1^2)$$

$$\dots = (kla_1, k(lb_1 + \frac{1}{2}k(k-1)a_1^2) + \frac{1}{2}k(k-1)(la_1)^2)$$

$$= (kla_1 + klb + \frac{1}{2}kla_1^2(l-1+(k-1)))$$

$$= (kla_1 + klb + \frac{1}{2}kl((k-1)a_1^2))$$

$$= kl \circ \alpha$$

$$(7)(k+l) \circ \alpha = ((k+l) a_1, (k+l)b_1 + \frac{1}{2}(k+l)(k+l-1)a_1^2)$$

$$= ((k+l)a_1, (k+l)b_1 + \frac{1}{2}(k^2 + l^2 + 2kl - k - l)a_1^2)$$

$$= (ka_1 + la_1, kb_1 + \frac{1}{2}k(k-1)a_1^2 + (b_1 + \frac{1}{2}l)(l-1)a_1^2 + ka_1 \cdot la_1)$$

$$= k \circ \alpha \oplus l \circ \alpha$$

(8)

$$k \circ (\alpha \oplus \beta) = k \circ (a_1 + a_2, b_1 + b_2 + a_1a_2) = (k(a_1 + a_2), k(b_1 + b_2 + a_1a_2) + \frac{1}{2}k(k-1)(a_1 + a_2)^2)$$

$$= (ka_1 + ka_2, kb_1 + \frac{1}{2}k(k-1)a_1^2 + kb_2 + \frac{1}{2}k(k-1)a_2^2 + ka_1a_2 + k(k-1)a_1a_2)$$

$$= (ka_1 + ka_2, (kb_1 + \frac{1}{2}k(k-1)a_1^2) + (kb_2 + \frac{1}{2}k(k-1)a_2^2 + (k^2a_1a_2)))$$

$$= (ka_1, kb_2 + \frac{1}{2}k(k-1)a_1^2) \oplus (ka_2, kb_2 + \frac{1}{2}k(k-1)a_2^2) = \alpha \oplus \beta$$

满足 3, 故 V 是一个线性空间

不成立。违反分配律,  $\forall \alpha \neq 0$ , 则会有  $\alpha = 2 \cdot \alpha = (1+1) \cdot \alpha = 1 \cdot \alpha + 1 \cdot \alpha = \alpha + \alpha$

⑥  $\Rightarrow \alpha = 0$ , 矛盾

$$P_{267,3} \textcircled{8} V = R^+ \quad P = R \quad a \oplus b = ab \quad k \circ a = a^k$$

解: V 非是①关于  $\oplus$  封闭②

任取  $a, b, c \in R^+, k, l \in R$

$$(1) a \oplus b = b \oplus a = ba$$

$$(2) (a \oplus b) \oplus c = (ab)c = a(bc) = a \oplus (b \oplus c)$$

$$(3) \text{零元 } 0 = 1, a \oplus 0 = a \cdot 1 = a$$

$$(4) \text{负元 } -a = a, a \oplus (-a) = a \cdot a = 1 = 0$$

$$(5) 1 \circ a = a^1 = a$$

$$(6) k \circ (l \circ a) = k \circ (a^l) = (a^l)^k = a^{lk} = (lk) \circ a$$

$$(7) (k+l) \circ a = a^{(k+l)} = a^k \cdot a^l = a^k \oplus a^l = k \circ a \oplus l \circ a$$

$$(8) k \circ (a \oplus b) = k \circ (ab) = (ab)^k = a^k b^k \\ = a^k \oplus b^k = k \circ a \oplus k \circ b$$

都成立, 故  $R^+$  关于  $\oplus$  做成  $R$  上的向量空间

$$P_{268,4} \textcircled{1} k0 = 0$$

$$k0 = \alpha, \text{ 则 } \alpha = k0 = k(0+0) = k0 + k0 = \alpha + \alpha$$

证: 设  $\therefore \alpha = \alpha + (-\alpha) = 0$

即  $k0=0$

$$4 \textcircled{2} k(\alpha - \beta) = k\alpha - k\beta$$

$$\therefore 0 = \alpha + (-\alpha) = \alpha + (-1)\alpha = [1 + (-1)] \cdot \alpha = 0 \cdot \alpha = 0$$

$$\therefore (-1)\alpha = -\alpha$$

$$\text{故 } k(\alpha - \beta) = k(\alpha + (-1)\beta) = k\alpha + k(-1)\beta = k\alpha + (-1)(k\beta)$$

$$= k\alpha + (-(k\beta)) = k\alpha - k\beta$$

**P<sub>268,5</sub>** 实函数空间F中, **0** 是 **0** 函数  $0(x), \forall x \in \text{定义域}$   $0(x)=0$ ,

于是  $k \cdot 1 + l \cdot \cos^2 t + m \cdot \cos 2t$

$$= k \cdot 1 + l \cos^2 t + m(2 \cos^2 t - 1)$$

$$= (k - m) \cdot 1 + (l + 2m) \cos^2 t$$

可取,  $m=1, k=1, l=-2$ , 则

$$1 \cdot 1 + (-2) \cos^2 t + 1 \cdot \cos 2t = 0 \cdot (x)$$

$\therefore 1 \cdot \cos^2 t, \cos 2t$  线性相关

**P<sub>268,6</sub>** 在  $P[x]$  中, **0** 元是 **0** 多项式(即系数全为 **0** 的多项式)

证:  $\because (f_1, f_2, f_3) = 1, (f_1, f_2) \neq 1, (f_2, f_3) \neq 1, (f_2, f_1) \neq 1,$

设  $a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) = 0$ , 不妨设  $a_1 \neq 0$

$$\therefore f_1(x) - \left(-\frac{a_2}{a_1}\right) f_2(x) + \left(-\frac{a_3}{a_1}\right) f_3(x)$$

$$\because (f_2, f_3) \neq 1, \text{ 被 } (f_2(x), f_3(x)) = d(x),$$

那么  $d(x)$  整除  $f_2, f_3$  的组合, 故  $d(x) \mid f_1(x)$ , 于是有

$$d(x) \mid (f_1(x), f_2(x), f_3(x))$$

与  $(f_1, f_2, f_3) = 1$  矛盾!

$$P_{268,7} \textcircled{1} \varepsilon_1 = (1, 1, 1, 1), \varepsilon_2 = (1, 1, -1, -1), \varepsilon_3 = (1, -1, 1, -1), \varepsilon_4 = (1, -1, -1, 1), \xi = (1, 2, 1, 1)$$

设  $\xi = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 + x_4 \varepsilon_4$  得方程解

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \because A^2 = 4E \\ \therefore A^{-1} = \frac{1}{4}A \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 5 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5/4 \\ 1/4 \\ -1/4 \\ -1/4 \end{pmatrix}$$

坐标为  $\left(\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$

$P_{268}.7.(2)\varepsilon_1 = (1, 1, 0, 1), \varepsilon_2 = (2, 1, 3, 1), \varepsilon_3 = (1, 1, 0, 0), \varepsilon_4 = (0, 1, -1, -1), \xi = (0, 0, 0, 1)$

设  $\xi = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4$ , 得

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + x_4 = 0 \\ x_1 + x_2 = x_4 = 0 \end{cases} \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

唯一解得  $x_1 = 1, x_2 = 0, x_3 = -1, x_4 = 0$

$\therefore \xi = \varepsilon_1 - \varepsilon_3$

在此基下的 坐标为  $(1, 0, -1, 0)$

$P_{268}.8① P^{n \times n}$  的一组是  $E_{ij}, i, j = 1, 2, \dots, n$ , 共有  $n^2$  个 (矩阵) 元素

$$\because \sum_{i,j=1}^n a_{ij} E_{ij} = 0 \Rightarrow A = (a_{ij}) = 0 \Rightarrow \forall i, j, a_{ij} = 0$$

它们线性无关

$$B = (b_{ij}) \in P^{n \times n}, \text{ 则 } B = \sum_{i,j=1}^n b_{ij} E_{ij}$$

且任何

$\dim P^{n \times n} = n^2$ , 它的一个基是  $E_{ij}, i, j = 1, 2, \dots, n$

$8② P^{n \times n}$  中全体对称矩阵集合  $S(P)$ , 它的一个基是  $E_{ij} + E_{ji}, i \leq j$

$$\dim S(P) = \frac{1}{2} n(n+1)$$

$P^{n \times n}$  中全体对称矩阵集合  $K(P)$ , 它的一个基是  $E_{ij} - E_{ji}, i < j$

$$\dim K(P) = \frac{1}{2} n(n-1)$$

$P^{n \times n}$  中全体上三角矩阵集合  $U(T)$ , 它的一个基是  $E_{ij}, i \leq j$

$$\dim U(T) = \frac{1}{2} n(n+1)$$



$P^{n \times n}$  中全体真下三角矩阵集合  $D^+(T)$ , 它的一个基是  $E_{ij}, i > j$

$$\dim D(T) = \frac{1}{2}n(n-1)$$

8②中,  $\theta$ , 零元是1, 取一个  $a > 0, a \neq 1$ , 则  $a \in \mathbb{R}^+$

那么  $\forall b \in \mathbb{R}^+$ , 取  $k = \log_a b (\because b = k \circ a = a^k \Rightarrow \lg b = k \cdot \lg a$

$$\therefore b = (\log_a b) \circ a = a^{\log_a b}$$

所以  $a$  是  $\mathbb{R}^+$  的一个基  $\dim_{\mathbb{R}} \mathbb{R}^+ = 1$

$$P_{268}.8(4), V = \left\{ f(A) \mid f(x) \in R[x], A = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}, \omega = \frac{-1 + \sqrt{3}i}{2} \right\}$$

解: 因为  $\omega^3 = 1$

$$A^2 = \begin{pmatrix} 1 & & \\ & \omega^2 & \\ & & \omega^4 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \omega^2 & \\ & & \omega \end{pmatrix} \cdot A^3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = E$$

所以

故任设  $f(x) \in R[x], f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

则  $f(A) = (a_0 + a_3 + a_6 + \cdots)E + (a_1 + a_4 + a_7 + \cdots)A + (a_2 + a_5 + a_8 + \cdots)A^2$

故  $f(A) = b_0E + b_1A + b_2A^2$

$\therefore E, A, A^2$  可表示  $V$  中所有元素。

$$xE + yA + zA^2 = 0 \Rightarrow \begin{cases} x + y + z = 0 \\ x + \omega^1 y + \omega^2 z = 0 \\ x + \omega^2 y + \omega z = 0 \end{cases}$$

如果

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3(\omega^2 - \omega) \neq 0, \text{ 所以 } x = y = z = 0 \text{ 只有零解}$$

$\therefore$  系数行列式

即,  $E, A, A^2$  线性无关, 由定理 1

$\dim V = 3$ , 它的一个基是  $E, A, A^2$

P269.9①  $\varepsilon_1 = (1, 0, 0, 0), \varepsilon_2 = (0, 1, 0, 0), \varepsilon_3 = (0, 0, 1, 0), \varepsilon_4 = (0, 0, 0, 1), \xi = (x_1, x_2, x_3, x_4)$

$$\eta_1 = (2, 1, -1, 1), \eta_2 = (0, 3, 1, 0), \eta_3 = (5, 3, 2, 1), \eta_4 = (6, 6, 1, 3),$$

解  $\xi = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4$

$$\therefore \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rightarrow \eta_1, \eta_2, \eta_3, \eta_4, \quad y = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)Ay$$

$$y = A^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4/9 & 1/3 & -1 & -1/9 \\ 1/27 & 4/9 & 1/3 & -23/27 \\ 1/3 & 0 & 0 & -2/3 \\ -7/27 & -1/9 & 1/3 & -6/27 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$\therefore$  坐标  $y$  为

( $A^{-1}$  计算附下页)

$$(A, E) \rightarrow \begin{pmatrix} 2 & 0 & 5 & 6 \\ 1 & 3 & 3 & 6 \\ -1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -7 & -9 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & -9 \\ 0 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & -3 & -7 \\ -1 & 0 & 1 & 3 \\ 2 & 1 & -3 & -8 \\ 1 & 0 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -27 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & -3 & -7 \\ -1 & 0 & 1 & 3 \\ -2 & -1 & 3 & 8 \\ 7 & 3 & -9 & -26 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 4/9 & 1/3 & -1 & -11/9 \\ 1/27 & 4/7 & -23/27 & -23/27 \\ 1/3 & 0 & -2/3 & -2/3 \\ -7/29 & -1/9 & \frac{1}{3} & 26/27 \end{pmatrix}$$

P<sub>269</sub>. 9. (2) 求由  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rightarrow \eta_1, \eta_2, \eta_3, \eta_4$  的过渡矩阵, 并求多在  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$  下的坐标.

$$\varepsilon_1 = (1, 2, -1, 0)$$

$$\eta_1 = (2, 1, 0, 1)$$

$$\varepsilon_2 = (1, -1, 1, 1)$$

$$\eta_2 = (0, 1, 2, 2)$$

$$\varepsilon_3 = (-1, 2, 1, 1)$$

$$\eta_3 = (-2, 1, 1, 2)$$

$$\varepsilon_4 = (-1, -1, 0, 1)$$

$$\eta_4 = (1, 3, 1, 2)$$

$$\xi = (1, 0, 0, 0)$$

解  $\because (\eta_1, \eta_2, \eta_3, \eta_4) = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)T$  因此.

$$(\eta_1, \eta_2, \eta_3, \eta_4, \xi) = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)(T, X)$$

$$\therefore (T, X) = A^{-1}B$$

$$\therefore B = \begin{pmatrix} 2 & 0 & -2 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 & 0 \end{pmatrix} = A \begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & -1 & 2 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} (T, X)$$

$$\therefore \left( \begin{array}{cccc|cccc} 1 & 1 & -1 & -1 & 2 & 0 & -2 & 1 & 1 \\ 2 & -1 & -2 & -1 & 1 & 1 & 1 & 3 & 0 \\ -1 & 1 & 1 & 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 1 & -1 & -1 & 2 & 0 & -2 & 1 & 1 \\ 0 & -3 & 4 & 1 & -3 & 1 & 5 & 1 & -2 \\ 0 & 2 & 0 & -1 & 2 & 2 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 0 \end{array} \right) ?$$

$$\rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & -2 & -2 & 1 & -2 & -4 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 0 \\ 0 & 0 & -2 & -3 & 1 & -2 & -5 & -2 & 1 \\ 0 & 0 & 7 & 4 & 0 & 7 & 11 & 7 & -2 \\ 0 & 0 & 1 & -5 & 0 & 1 & -4 & 1 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & -12 & 1 & 0 & -12 & 1 & 3 \\ 0 & 1 & 0 & 6 & 1 & 1 & 6 & 1 & -1 \\ 0 & 0 & 1 & -5 & 0 & 1 & -4 & 1 & 1 \\ 0 & 0 & 0 & -13 & 0 & 0 & -13 & 0 & 3 \end{array} \right)$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 3/13 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 5/13 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -2/13 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3/13 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

因此: 过渡矩阵

令 在  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$  下的坐标为  $\left( \frac{5}{13}, \frac{5}{13}, -\frac{2}{13}, -\frac{3}{13} \right)$

p269.9③  $\varepsilon_1 = (1, 1, 1, 1), \varepsilon_2 = (1, 1, -1, -1), \varepsilon_3 = (1, -1, 1, -1), \varepsilon_4 = (1, -1, -1, 1)$

$$\eta_1 = (1, 1, 0, 1), \eta_2 = (2, 1, 3, 1), \eta_3 = (1, 1, 0, 0), \eta_4 = (1, -1, -1, 1)$$

求  $\xi = (1, 0, 0, -1)$  在  $\eta_1, \eta_2, \eta_3, \eta_4$  下的坐标

解 若  $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)T = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$  那么

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & -1 \\ 1 & 1 & 0 & -1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & -1 \\ 1 & 1 & 0 & -1 \end{pmatrix}$$

$$\frac{1}{4} \begin{pmatrix} 3 & -1 & 2 & -1 \\ 1 & -1 & 2 & 3 \\ -1 & 3 & 0 & -1 \\ 1 & -1 & 0 & -1 \end{pmatrix} \text{ 而 } \xi = (\eta_1, \eta_2, \eta_3, \eta_4)y = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)x$$

$$\begin{aligned} &= (\eta_1, \eta_2, \eta_3, \eta_4)TY \\ \therefore Y = T^{-1}X \quad X = A^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} &= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

不如直接解出

$$\xi = (\eta_1, \eta_2, \eta_3, \eta_4)y \quad \therefore \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 1 & 1 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

y

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -3 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -2 \\ 1 \\ 1 \\ -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ -1/2 \\ 4 \\ -3/2 \end{pmatrix}$$

故  $\xi = -2\eta_1 - \frac{1}{2}\eta_2 + 4\eta_3 - \frac{3}{2}\eta_4$  在该基下坐标为  $\left(-2, -\frac{1}{2}, 4, -\frac{3}{2}\right)$

$$\xi = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)x = (\eta_1, \eta_2, \eta_3, \eta_4)X \quad \text{作 } A = \begin{pmatrix} 2 & 0 & 5 & 6 \\ 1 & 3 & 3 & 6 \\ -1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$$

P269.10. 设

$$\begin{pmatrix} 1 & 0 & 5 & 6 \\ 1 & 2 & 3 & 6 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \therefore$$

$$x = k \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

解

只要  $k \neq 0$  即可, 取  $k=1$  即有  $\xi = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4 = (1, 1, 1, -1)$

P269.11,  $V_1 = R, V_2 = R^+_R$  规定  $B: R \rightarrow R^+, a \rightarrow e^a$  (自然对数)

则  $B$  是  $1-1$  的和映上以  $(a \neq b \Rightarrow e^a \neq e^b, \text{正定 } b \text{ 原为 } \ln b)$

又  $\because \delta(a+b) = e^{a+b} = e^a \cdot e^b = e^a \oplus e^b = \delta(a) \oplus \delta(b)$

$$\delta(ka) = e^{ka} = (e^a)^k = k \cdot e^a = k \cdot \delta(a)$$

故  $\delta$  就是同构,  $R \cong R^+$  其实任取一个数  $d(d>0, d \neq e)$  代替  $e$  均可:

(p269)12 设  $V_1 \subseteq V_2, V_1 \leq V, V_2 \leq V$ , 且  $\dim V_1 = \dim V_2$

证明  $V_1 = V_2$

只须证  $V_1 \supseteq V_2$

证: 设  $\dim V_1 = \dim V_2 = r$ , 且  $\alpha_1, \alpha_2, \dots, \alpha_r$  为  $V_1$  的个基任取  $\beta \in V_2$

$\alpha_1, \alpha_2, \dots, \alpha_r \in V_2$  且在  $V_2$  中线性无关.

因为  $\dim V_2 = r$ , 故  $V_2$  中这  $r+1$  个向量  $\beta, \alpha_1, \alpha_2, \dots, \alpha_r$ , 线性相关, 由临界定理.

$$\beta \leftarrow \alpha_1, \alpha_2, \dots, \alpha_r \Rightarrow \beta \in V_1$$

即  $V_2 \subseteq V_1$  中 (得证)

$$(P_{269}, 13), C(A) = \{x \in P^{n \times n} \mid AX = XA\} \subseteq P^{n \times n}$$

$$1) \because 0 \in C(A) \neq \phi$$

$$\begin{aligned} \forall X, Y \in C(A) &\Rightarrow A(X+Y) = AX + AY = XA + YA = (X+Y)A \\ &\Rightarrow A(kX) = k(AX) = k(XA) = X(kA) \end{aligned}$$

$$\therefore x+y, kA \in C(A) \text{ 即 } C(A) \leq P^{n \times n}$$

$$2) \because \forall x \in P^{n \times n}, \text{ 有 } XE = EX, \text{ 故 } X \in C(E)$$

$$\therefore P^{n \times n} \subseteq C(E) \text{ 但 } C(E) \subseteq P^{n \times n}$$

$$\therefore \text{当 } A=E \text{ 时, } C(A) = C(E) = P^{n \times n}$$

$$3) \quad \forall x = (x_{ij}) \in P^{n \times n} \quad \text{由于} \quad A = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \cdots & \\ & & & n \end{pmatrix}$$

$$\text{有 } X \in C(A) \Leftrightarrow XA = AX$$

$$\Leftrightarrow XA \text{ 第 } i \text{ 行 } j \text{ 列元素, } jx_{ij} = AX \text{ 第 } i \text{ 行 } j \text{ 列元素 } ix_{ij},$$

$$(\forall, ij)$$

$$\Leftrightarrow (\forall i, j), \quad xi_j(i-j) = 0$$

$$\Leftrightarrow i \neq j \text{ 时 } xi_j = 0, \text{ 若 } i = j, \text{ 则 } i = j, \text{ 则 } x_{ii} \text{ 任意}$$

$$\Leftrightarrow X = \begin{pmatrix} x_{11} & & & \\ & x_{22} & & \\ & & \cdots & \\ & & & x_{nn} \end{pmatrix} = \sum_{i=1}^n x_{ii} E_{ii}$$

$E_{11}, E_{22}, \dots, E_{nn}$  线性无关

此时  $C(A)$  是全体对角矩阵,  $E_{11}, E_{22}, \dots, E_{nn}$  是它的一个基, 故  $\dim C(A) = n$

$$P270.14 \text{ 设 } x = \begin{pmatrix} a & b & c \\ d & e & f \\ q & h & i \end{pmatrix} \quad Ax = \begin{pmatrix} a & b & c \\ b & e & f \\ 3a+d+2g & 3b+e+2h & 3c+f+2i \end{pmatrix}$$

$$XA = \begin{pmatrix} a+3c & b+c & 2c \\ d+3f & e+f & 2f \\ q+3i & h+i & 2i \end{pmatrix} \quad \therefore AX = XA \Rightarrow a = a+3c \Rightarrow c=0, d = d+3f \Rightarrow f=0$$

$$\text{且 } \begin{cases} 3a+d+2g = g+3i \\ 3b+e+2h = h+i \\ 2i = 2i \end{cases} \quad \therefore \begin{cases} 3a+d+q = 3i \\ 3b-e+h = i \\ i, a, d, b, e \text{ 任意} \end{cases}$$

$\therefore$  依次取  $(a, b, d, e, i) = \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$  得基元素

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

$\dim C(A) = 5$

p270.15  $c_1 c_3 \neq 0, c_1 d + c_2 \beta + c_3 r = 0 \Rightarrow \alpha = -\frac{c_2}{c_1} \beta - \frac{c_3}{c_1} r, r = \frac{c_1}{c_3} \alpha - \frac{c_2}{c_3} \beta$

$$\therefore \alpha \cdot \beta \xrightarrow{\beta} \beta \cdot r \Rightarrow L(\alpha \cdot \beta) = L(\beta \cdot r)$$

P270.16① 
$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & -3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \therefore \alpha_4, \alpha_2, \alpha_3 \text{ 线性无关}$$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 3$  故基为  $\alpha_2, \alpha_3, \alpha_4$ ,

P270.16② 
$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 & -1 \\ -1 & 1 & -3 & 1 \\ 4 & 5 & 3 & -1 \\ 1 & 5 & -3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 3 & -1 \\ 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 5 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 3 & -1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$  秩  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 2, \alpha_1, \alpha_2$  是一个极大无关组

$\therefore \dim L(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 2, \alpha_1, \alpha_2$  是它的一个基

P270.17 
$$\begin{pmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 3 & -8 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2/3 & -5/3 & 4/3 \\ 0 & 1 & -8/3 & 7/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1/9 & -2/9 \\ 0 & 1 & -8/3 & 7/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

系数矩阵为 A, 秩 (A) = 2 基础解系含  $4-2=2$  个向量, 可为 
$$\begin{pmatrix} -1 \\ 24 \\ 9 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ -21 \\ 0 \\ 9 \end{pmatrix}$$

∴解空间的维数为 2, 基底一个是

$$(-1, 24, 9, 0), (2, 21, 0, 9)$$

(P270, 18, ①) 解设  $V_1 = L(\alpha_1, \alpha_2)$   $V_2 = L(\beta_1, \beta_2)$

若设  $r = x_1\alpha_1 + x_2\alpha_2 = x_3\beta_1 + x_4\beta_2$  即  $r \in V_1 \cap V_2$

$$\begin{cases} x_1 - x_2 - 2x_3 - x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - 3x_4 = 0 \\ x_2 - x_3 - 7x_4 = 0 \end{cases} \begin{pmatrix} 1 & -1 & -2 & -1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & -3 \\ 0 & 1 & -1 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -2 & -1 \\ 0 & 3 & 5 & 3 \\ 0 & 2 & 2 & -2 \\ 0 & 1 & -1 & -7 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -2 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -3 \\ 1 \end{pmatrix}$$

有非零解如  $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -3 \\ 1 \end{pmatrix}$  即

∵秩 $(\alpha_1, \alpha_2, \beta_1, \beta_2) = 3$  所以  $\dim(V_1 \cap V_2) = 2 + 2 - 3 = 1$  维

它的一个基是  $r = (-5, 2, 3, 4)$

(P270, 18, ②) 解: 设  $V_1 = L(\alpha_1, \alpha_2)$ ,  $V_2 = L(\beta_1, \beta_2)$  则由

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

由此秩 $(\alpha_1, \alpha_2, \beta_1, \beta_2) = 4$  ∴  $\dim(V_1 + V_2) = \dim L(\alpha_1, \alpha_2, \beta_1, \beta_2) = 4$  而

$\dim V_1 = \dim V_2 = 2$ , 所以  $\dim(V_1 \cap V_2) = 2 + 2 - 4 = 0$

此时  $V_1 \cap V_2$  没有基.

(P270, 18, ③) 解: 设  $V_1 = L(\alpha_1, \alpha_2, \alpha_3)$ ,  $V_2 = L(\beta_1, \beta_2)$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & +2 & -1 & -2 \\ 3 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 4 & -2 \\ 0 & 2 & 0 & -3 \end{pmatrix} \quad \begin{matrix} \therefore \dim V_1 = 3 \\ \text{秩为 } 3 \end{matrix}$$

由

显有  $\dim V_2 = 2$

设  $r = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = x_4\beta_2 \in V_1 \cap V_2$  则得

$$\begin{cases} x_1 + 3x_2 - x_3 - 2x_4 + x_5 = 0 \\ 2x_1 + x_2 - 5x_4 - 2x_5 = 0 \\ -x_1 + x_2 + x_3 + 6x_4 + 7x_5 = 0 \\ -2x_1 + x_2 - x_3 + 5x_4 - 3x_5 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ -1 & 1 & 1 & 7 \\ -2 & 1 & 1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & -5 & 2 & -4 \\ 0 & 4 & 0 & 8 \\ 0 & 7 & -3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 5 & 13 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 11 & 27 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{秩为 } 3: \text{即秩}(\alpha'_1, \alpha'_2, \alpha'_3, \beta'_1, \beta'_2) = 4$$

$$\therefore \dim(V_1 + V_2) = 4 \text{ 故 } \dim(V_1 \cap V_2) = 5 - 4 = 1$$

取方程组一个非零解  $(x_1, x_2, x_3, x_4, x_5) = (3, -1, -2, 1, 0)$

即  $\beta_1 = r = 3\alpha_1 - \alpha_2 - 2\alpha_3 \in V_1 \cap V_2$  是一个所求的基

P270.19  $x_1 + x_2 + \cdots + x_n = 0$  的空间  $V_1$   $\dim V_1 = n - 1$  秩  $(1, 1, 1, \dots, 1) = n - 1$

$$x_1 = x_2 = \cdots = x_n = 0 \text{ 的解空间 } V_2, \quad A = \begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & \ddots \\ & & & 1 & -1 \end{pmatrix}$$

$x_1 = x_2 = \cdots = x_n$  的解空间  $V_2$

$$\dim V_2 = n - \text{秩}(A) = n - (n - 1) = 1$$

$$\left. \begin{aligned} \xi = (\alpha_1, \alpha_2, \dots, \alpha_n) \in V_1 \cap V_2, \text{ 则 } & \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha \\ & \Rightarrow \alpha_1, \alpha_2 + \cdots + \alpha_n (= n\alpha) = 0 \end{aligned} \right\} \Rightarrow \alpha = 0$$

若

$\therefore \xi = (\alpha, \alpha, \dots, \alpha) = 0$  即由定理8推论  $V_1 + V_2$  是直和

$$\therefore \dim(V_1 \oplus V_2) = (n - 1) + 1 = n \quad \text{是 } V_1 + V_2 \subseteq P^n,$$

$$\dim(V_1 \oplus V_2) = \dim P^n \text{ (由12题6.88.8.3) } P^n = V_1 \oplus V_2$$

P271, 20, 设  $V = V_1 \oplus V_2, V_1 = V_{11} \oplus V_{12}$  那么  $V = V_{11} \oplus V_{12} \oplus V_2$

证一: 设  $0 = \alpha_{11} + \alpha_{12} + \alpha_2 \because V = V_1 \oplus V_2$

$$\therefore \alpha_2 = 0, \alpha_{11} + \alpha_{12} \in V_1 \text{ 也为 } 0 \text{ 即}$$



$O = \alpha_{11} + \alpha_{12}$  为  $V_{11} \oplus V_{12}$  的直和分解或故

$\alpha_{11} = 0, \alpha_{12} = 0$ , 所以 0 有唯一分解式  $0 = 0_{11} + 0_{12} + 0_2$

$$\therefore V = V_{11} \oplus V_{12} \oplus V_2$$

证二:  $\dim V = \dim V_1 + \dim V_2 = (\dim V_{11} + \dim V_{12}) + \dim V_2$

证毕

P271.21  $\because V = L(\alpha_1, \alpha_2, \dots, \alpha_m)$  作  $W_i = L(\alpha_i)$

若  $\forall \beta \in V \quad \beta = \sum_{i=1}^n \beta_i = \sum_{i=1}^n r_i \quad \beta_i V_i \in W_i$  可设  $\beta = b_i \alpha_i, y_i = -\frac{c_2}{c_3} \beta$

$\Rightarrow 0 = \Sigma(\beta_i - r_i) = \Sigma(b_i - c_i) d_i$ , 无关性  $\Rightarrow b_i - c_i = 0 \therefore b_i = c_i$

因此  $\beta_i = r_i \quad \therefore \beta$  的分解式唯一,  $\therefore V = W_1 \oplus W_2 \oplus \dots \oplus W_n$

P271.22  $\because W_i = \sum_{j=1}^{i-1} V_j \subseteq \sum_{j \neq i}^s V_j = V_i$  故若  $\sum_{i=1}^s V_i$  为直和则  $r_i \cap V_i = \{0\}$

$\therefore V_i \cap W_i = \{0\}$  从而必要性显然.

反过来证充分性

$\sum_{i=1}^s V_i$  不是直和, 有  $\alpha_1, \alpha_2, \dots, \alpha_s, \alpha_i \in V_i$  不全为 0, 且  $0 = \alpha_1 + \alpha_2 + \dots + \alpha_s$

$\alpha_k$  为  $\alpha_s, \alpha_{s-1}, \dots, \alpha_2, \alpha_1$  中第一个不为 0 的向量故

$$0 \neq \alpha_k = (-\alpha_1) + (-\alpha_2) + \dots + (-\alpha_{k-1}) \subseteq \sum_{j=1}^{k-1} V_j = W_k$$

显然若  $k=1 \Rightarrow \alpha_1 \neq 0, \alpha_2 = \dots = \alpha_s = 0$ , 而  $0 - \alpha_1 + 0 + 0 + \dots + 0$  矛盾  $\therefore k \geq 2$

又  $\alpha_k \in V_k$  从而  $V_k \cap W_k \neq \{0\}$  与已知矛盾, 故

$$\sum_{i=1}^s V_i = V_1 \oplus V_2 \oplus \dots \oplus V_s$$

P271.23②当平面经过原点是线性子空间, 不经过原点则不是

$\because$  若  $0 \in$  平面  $\alpha \in$  平面

则  $k\alpha (k \neq 1) \notin$  平面

23② $L_1+L_2$ 生成直线(当 $L_1=L_2$ )

生成直线(当 $L_1 \neq L_2$ )

$L_1+L_2+L_3$ 生成直线(当 $L_1=L_2+L_3$ )

生成直线(当 $L_1, L_2, L_3$ , 共面)

生成空间 $\mathbb{R}^3$ (当 $L_1, L_2, L_3$ 不共面)

23②当然不一定有 如右图

$V+V$  不x平面 y为平面中线

$$y \subseteq x$$

$$\text{但 } y \cap V = 0, Y \cap V = 0, \therefore y \neq (y \cap V) + (y \cap V) = 0$$

P<sub>271</sub> 补 1①因为  $f_i(x) = \frac{f(x)}{x-a_i}$   $f(x) = (x-a_1)(x-a_2)\dots(x-a_n)$

$$f_i(a_k) = 0 (k \neq i) \quad \text{或} \quad f_i(a_i) = \prod_{(j \neq i)} (a_i - a_j)$$

如果 $n=1$ , 则  $f_1(x)=1$  显然( $\neq 0$ ) 线性无关

如果 $n \geq 2$ , 而  $f_1(x), f_2(x), \dots, f_n(x)$  线性相关, 则不妨设  $f_1(x) = \sum_{i=2}^n k_i f_i(x)$

但是在  $x = a_1$  处值, 右边恒为 0, 左边为  $f_1(a_1) = \prod_{j=2}^n (a_1 - a_j) \neq 0$

矛盾  $\therefore f_1(x), f_2(x), \dots, f_n(x)$  线性无关, 而  $\dim P[X] = n$

以及单个  $f_i(x)$ , 次数  $\leq n-1, \therefore f_i(x) \in p[x]_n$ . 故诸  $f_i(x)$  形成基

P<sub>271</sub> 补 1②  $x^n = 1$  的单数根为  $\omega, \omega^2, \omega^3, \dots, \omega^n = 1$ , ( $\omega$  本原的)  $a_i = \omega^i$

$$\therefore f_i(x) = \frac{f(x)}{x-a_i} = \frac{x^{n-1}}{x-\omega^i} = \frac{\omega^u - (\omega^i)^u}{x-\omega^i} = x^{n-1} + \omega^i x^{u-2} + \omega^i x^{u-3} + \dots + \omega^{(n-1)i}$$

$$(f_1(x), f_2(x), \dots, f_n(x)) = (1, x, x^2, \dots, x^{n-1}) \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \omega & \omega^2 & \dots & \omega^{n-1} & 1 \\ \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)^2} & 1 \end{pmatrix}$$

$$\therefore T = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \omega & \omega^2 & \cdots & \omega^{n-1} & 1 \\ \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} & 1 \end{pmatrix}$$

(P271 补 2) 因  $\alpha_1, \alpha_2, \cdots, \alpha_n$  线性无关, 设秩  $(A) = r, \exists P, Q$  可逆使

$$A = P \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}_{n \times s} Q$$

所以

$$\begin{aligned} & (\beta_1, \beta_2, \cdots, \beta_s) = (\alpha_1, \alpha_2, \cdots, \alpha_n) A \\ & = (\alpha_1, \alpha_2, \cdots, \alpha_n) P \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} Q = (r_1, r_2, \cdots, r_n) \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} Q = (r_1, r_2, \cdots, r_n) \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} Q \\ & = (r_1, r_2, \cdots, r_r, 0, \cdots, 0) \end{aligned}$$

$$\therefore \beta_1, \beta_2, \cdots, \beta_s \leftarrow r_1, r_2, \cdots, r_r \text{ 但 } Q \text{ 可逆 } (r_1, \cdots, r_r, 0, \cdots, 0) = (\beta_1, \cdots, \beta_s) Q^{-1}$$

$$\therefore r_1, r_2, \cdots, r_r \leftarrow \beta_1, \beta_2, \cdots, \beta_s \text{ 由定理了}$$

$$\dim(L(\beta_1, \beta_2, \cdots, \beta_s)) = \dim(L(r_1, r_2, \cdots, r_r)) = \text{秩}(r_1, r_2, \cdots, r_r) = r = \text{秩}(A)$$

P271 补 3, 设  $f(x_1, x_2, \cdots, x_m) = x'AX$

由秩  $(f) = n, f$  的符号差为  $S$ , 那么  $f$  的惯性指数  $p = \frac{n+s}{2}$

$$f \text{ 的负惯性指数 } q = \frac{1}{2}(n-s)$$

非退化线性替换, 使

$$f(x_1 \cdots x_n) = g(y_1 \cdots y_n) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2$$

作  $n$  维向量

$$y_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \varepsilon_1 + \varepsilon_{p+1}, y_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \varepsilon_2 + \varepsilon_{p+2}, \cdots, y_q = \varepsilon_q + \varepsilon_{p+q} = \varepsilon_q + \varepsilon_n$$

那么  $g(y_1, \dots, y_n)$  在  $y_i$  处的值为 0 且, 若

$$y_o = b_1 y_1 + b_2 y_2 + \dots + b_q y_q$$

$$\text{则 } g(y_1, \dots, y_n) \text{ 在 } y_o \text{ 的值为 } b_1^2 + b_2^2 + \dots + b_q^2 - b_1^2 - b_q^2 = 0$$

于是  $V_2 = L(y_1, y_2, \dots, y_q)$  及  $V_1 = L(cy_1, cy_2, \dots, cy_q)$

$$\therefore f \text{ 在 } V_1 \text{ 中任意处 } \sum_{i=1}^q q_i(cy_i) \text{ 的值, 等于 } q \text{ 在 } \sum_{i=1}^q a_i y_i \text{ 的值为 } 0$$

$\therefore y_1, y_2, \dots, y_q$  线性无关,  $\therefore x_1 = cy_1, x_2 = cy_2, \dots, x_q = cy_q$  线性无关

$$\therefore \dim V_1 = q = \frac{1}{2}(n-s) = \frac{1}{2}(n-s/)$$

b°如果  $s < 0$  则  $p < q$  作  $x = cy, f = g(y_1, \dots, y_n)$  为规范形

这对可取  $y_1, y_2, \dots, y_p$  生成  $V_2$  且  $V_2$  要化  $g$

$$\text{同时作 } V_1 = g(x_1, x_2, \dots, x_p) = L(y_1, y_2, cy_p)$$

(同 a°)  $V_1$  即使得  $f/V_1 = 0$  且

$$\dim V_1 = P = \frac{1}{2}(n+s) = \frac{1}{2}(n-s/)$$

(P271补4)证法一:  $\therefore V_1 \neq V, V_2 \neq V$

故若  $V_1 \subseteq V_2$  则取一  $\alpha \notin V_2$  ( $\alpha$  必存在) 即可

若  $V_1 \supseteq V_2$  则任取一  $\alpha \notin V_1$  ( $\alpha$  也存在) 即可

若  $V_1 \not\subseteq V_2, V_2 \not\subseteq V_1$  则可取  $\alpha \in V_1, \alpha \notin V_2$  和  $\beta \in V_2, \beta \notin V_1$

$\therefore \alpha + \beta \in V_2 \Rightarrow \beta \in V_1$  矛盾,  $\alpha + \beta \in V_2$  也矛盾  $\therefore \alpha + \beta \notin V_1, \notin V_2$  即为所求.

证法 2: 若  $V_1 \subseteq V_2$  或  $V_2 \subseteq V_1$ , 同上显然

当  $V_1 \subseteq V_2, V_2 \not\subseteq V_1$  时, 取  $\alpha \notin V_1, \beta \in V_1, \beta \in V_2$ , 考虑一切的  $\alpha + k\beta$  如右图

$\{\alpha + k\beta\}$  理解为  $V_1$  的平行体.

断言 (a) 若有  $\alpha \in P, \alpha + k\beta \in V_1$  (b) 至多存在一个  $k$ , 使  $\alpha + k\beta \in V_2$

证 (a) 有  $k \in p, \alpha + k\beta \in V_1 \therefore \beta \in V_1 \Rightarrow \alpha = (\alpha + k\beta) - k\beta \in V_1$ , 矛盾!

(b) 若有  $k_1 \neq k_2 \in p$  使  $\alpha + k_1\beta \in V_2, \alpha + k_2\beta \in V_2$  则,  $k_1\beta - k_2\beta \in V_2$

$$\Rightarrow (k_1 - k_2)\beta \in V_2 \Rightarrow \beta \in V_2 \text{ 矛盾!}$$

故若  $k_o \in p, \alpha + k_o\beta \notin V_2$ , 则  $\alpha + k_o\beta$  即为所求,  $\notin V_1, \notin V_2$ ,

若  $\alpha + k_o\beta \in V_2$  则任取  $k_1 \neq k_o$  有  $\alpha + k_1\beta \notin V_1, \notin V_2$  即为所求

证法二虽然思想复杂, 却可以把问题做大

(P272) 补 5

$s=1$  虽然( $\because V_1$  非平凡,  $\therefore V_1 \neq V$ )

$s=2$  命题已证, 即第 4 题设  $S=k$  时命题成交, 考虑  $s=k+1$  时,  $V_1, V_2, \dots, V_k, V_{k+1}$  皆非

平凡了空间对于  $V_1, V_2, \dots, V_k$  任取  $\alpha \notin V_{k+1}$  ( $\because V_{k+1} \neq V$ ) 考虑一切  $\alpha + k\beta$

同样 (类似 4 题证法  $(P_6, 92, 12, 3)$ ),  $\forall k \in p, \alpha + k\beta \in V_{k+1}$  对每个  $V_i (i=1, 2, \dots, k)$  至

多只须一个使,  $\alpha + k_i\beta \in V_i$

取  $r_o$  为不同于  $r_1, r_2, \dots, r_k, 1$  的任一数即

$$t_o \in p - \{t_1, t_2, \dots, V_k, V_{k+1}\}$$

那么  $t_o \notin V_1, V_2, \dots, V_k, V_{k+1} \therefore \alpha + t_o\beta$  即为所求

## 第九章 第九章 欧几里得空间习题解答

P394.1.1

$A$  正定  $\therefore (\alpha, \alpha) = \alpha A \alpha' \geq 0$  (" $=$ "  $\Leftrightarrow \alpha = 0$ ) 非负性证得

由矩阵失去, 线性性成立, 再由  $(\beta, \alpha) = \beta A \alpha' = (\beta A \alpha')' = \alpha' A' \beta = (\alpha, \beta)$   
对称性成立, 是一个内积

$$P394.1.2 \quad (\varepsilon_i, \varepsilon_j) = (0 \cdots \overset{i}{1} 0 \cdots 6) A \begin{pmatrix} 6 \\ 1 \\ 1 \\ 9 \end{pmatrix} = \alpha_{ij}$$

$\therefore \varepsilon_i, \varepsilon_j, \cdots \varepsilon_n$  的度量矩阵即为 A

$$P394.1.2 \quad |(\alpha, \beta)| \leq |\alpha| |\beta|$$

$$\because (\alpha, \beta) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i y_j$$

$$\therefore c-s-B \text{ 不等式为 } |(\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i y_j)| \leq \sqrt{(\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j)(\sum_{i=1}^n \sum_{j=1}^n a_{ij} y_i y_j)}$$

$$P393.2 \text{ ①, } \alpha=(2,1,3,2), \beta=(1,2,-2,1)$$

$$\therefore |\alpha| = \sqrt{18} = \sqrt[3]{2}, |\beta| = \sqrt{10}, (\alpha, \beta) = 0, \therefore \alpha \perp \beta$$

$$\therefore \langle \alpha, \beta \rangle = \frac{\pi}{2}$$

$$P393.2 \text{ ②, } \alpha=(1,2,2,3), \beta=(3,1,5,1)$$

$$|\alpha| = \sqrt{18} = \sqrt[3]{2}, |\beta| = \sqrt{36} = 6, (\alpha, \beta) = 18$$

$$\therefore (\alpha, \beta) = \arccos \frac{(\alpha, \beta)}{|\alpha| |\beta|} = \arccos \frac{18}{\sqrt[3]{2} \cdot 6} = \arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

$$P393.2 \text{ ③, } \alpha=(1,1,1,2), \beta=(3,1,-1,0)$$

$$|\alpha| = \sqrt{7} \quad |\beta| = \sqrt{11} \quad (\alpha, \beta) = 3$$

$$\therefore \langle \alpha, \beta \rangle = \arccos \frac{3}{\sqrt{77}} = 70^\circ 0' 30'' 38$$

$$P393.3 \quad \because |\alpha + \beta| \leq |\alpha| + |\beta|$$

$$\begin{aligned} \therefore d(\alpha, \gamma) &= |\alpha - \gamma| = |(\alpha - \beta) + (\beta - \gamma)| \leq |\alpha - \beta| + |\beta - \gamma| \\ &= d(\alpha, \beta) + d(\beta, \gamma) \end{aligned}$$

**P393.4** 在  $R^4$  中求一单位向量与  $(1, 1, -1), (1, -1, 1, 1), (2, 1, 1, 3)$  正交  
解设所求

$\alpha = (x_1, x_2, x_3, x_4)$  则  $\sum x_i^2 = 1$ , 且

$x$  与各向量的内积为 0 得

$$\begin{cases} x_1 + x_2 - x_3 + x_4 = 0 \\ x_1 - x_2 - x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + 3x_4 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 2 & 1 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & +1 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \text{ 令 } x_4 = 3, \text{ 得}$$

$$x_1 = 4, x_2 = 0, x_3 = -1$$

$$\alpha = \frac{1}{\pm\sqrt{26}}(-4, 0, -1, 3), \quad (\text{单位化})$$

P393.5 ①证: 因为  $(\gamma, \alpha_i) = 0, i = 1, 2, \dots, n$ , 而  $\alpha_1, \alpha_2, \dots, \alpha_n$  是一个基

$$\therefore (\gamma, \gamma) = (\gamma, \sum_{i=1}^n k_i \alpha_i) = \sum_{i=1}^n k_i (\gamma, \alpha_i) = 0.$$

因此, 必有  $\gamma = 0$ .

P393.5 ②证,  $\therefore (\gamma_1, \alpha_i) = (\gamma_2, \alpha_i), i = 1, 2, \dots, n$ ,

$$\therefore (\gamma_1 - \gamma_2, \alpha_i) = 0, i = 1, 2, \dots, n$$

由第 ① 小题:  $\gamma_1 - \gamma_2 = 0$ , 故  $\gamma_1 = \gamma_2$

$$\therefore (\alpha_1, \alpha_2, \alpha_3) = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \frac{1}{3} \begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{pmatrix}$$

P393.6

而  $\frac{1}{3} \begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{pmatrix}$  是正交矩阵, 所以  $\alpha_1, \alpha_2, \alpha_3$  是标准正交基

P393.7

$$\alpha_1 = \varepsilon_1 \varepsilon_s, \alpha_2 = \varepsilon_1 - \varepsilon_2 + \varepsilon_4 / \varepsilon_3 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$\text{解: } \beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1 = \alpha_2 - \frac{1}{2} \beta_1 = \frac{1}{2} \varepsilon_1 - \varepsilon_2 + \varepsilon_4 - \frac{1}{2} \varepsilon_5 = \frac{1}{2} (\varepsilon_1 - 2\varepsilon_2 + 2\varepsilon_4 - \varepsilon_5)$$

$$\beta_3 = \alpha_3 - \frac{2}{2} \beta_1 - \frac{1}{10} \beta_2 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_5$$

再正交化称:

$$\eta_1 = \frac{1}{\sqrt{2}} (\varepsilon_1 + \varepsilon_5)$$

$$\eta_2 = \frac{1}{\sqrt{10}} (\varepsilon_1 - 2\varepsilon_2 + 2\varepsilon_4 - \varepsilon_5)$$

$$\eta_3 = \frac{1}{2} (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_5)$$

$$\begin{pmatrix} 2 & 1 & -1 & 1 & -3 \\ 1 & 1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 & 5 \end{pmatrix} X = 0$$

P394.8, 解:

$$\eta_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \eta_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \eta_3 = \begin{pmatrix} 4 \\ -5 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

解出:

Schmidt:

$$\beta_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \beta_2 = \eta_2 - \frac{1}{2} \beta_1 = \frac{1}{2} \begin{pmatrix} -2 \\ 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \quad \beta_3 = \eta_3 + \frac{5}{2} \beta_1 + \frac{13}{10} \begin{pmatrix} -2 \\ 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 7 \\ -6 \\ 6 \\ 13 \\ 5 \end{pmatrix}$$

单位化便得到解空间的标准正交基:



$$\varepsilon_1 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \quad \varepsilon_2 = \begin{pmatrix} -2/\sqrt{10} \\ 1/\sqrt{10} \\ -1/\sqrt{10} \\ 2/\sqrt{10} \\ 0 \end{pmatrix} \quad \varepsilon_3 = \frac{1}{\sqrt{315}} = \begin{pmatrix} 7 \\ -6 \\ 6 \\ 13 \\ 5 \end{pmatrix}$$

**P394.9**  $(f, g) = \int_{-1}^1 f(x)g(x)dx$

已知  $\alpha_1 = 1, \alpha_2 = x, \alpha_3 = x^2, \alpha_4 = x^3$

解:  $\beta_1 = \alpha_1 = 1$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1 = x - \frac{\int_{-1}^1 x dx}{*} x$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} \beta_2 = x^2 - \frac{2}{2} \cdot 1 - 0 = x^2 - \frac{1}{3}$$

$$\beta_4 = \alpha_4 - \frac{(\alpha_4, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_4, \beta_2)}{(\beta_2, \beta_2)} \beta_2 - \frac{(\alpha_4, \beta_3)}{(\beta_3, \beta_3)} \beta_3 = x^3 - 0 - \frac{2}{2} x = x^3 - \frac{3}{5} x$$

$$\text{又} \because (\beta_1, \beta_1) = 2 \quad |\beta_1| = \sqrt{2}, \quad (\beta_2, \beta_2) = \frac{2}{3} \quad |\beta_2| = \frac{2}{\sqrt{6}}$$

$$(\beta_3, \beta_3) = \int_{-1}^1 (x^4 - \frac{2}{3}x^2 + \frac{1}{9})dx = \frac{8}{45} \quad |\beta_3| = \frac{4}{\sqrt{10}}$$

$$(\beta_4, \beta_4) = \int_{-1}^1 (x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2)dx = \frac{8}{175} \quad |\beta_4| = \frac{4}{\sqrt{14}}$$

单位化标准正交基

$$\gamma_1 = \frac{1}{\sqrt{2}}, \quad \gamma_2 = \frac{\sqrt{6}}{2}x, \quad \gamma_3 = \frac{\sqrt{10}}{4}(3x^2 - 1), \quad \gamma_4 = \frac{\sqrt{14}}{4}(5x^3 - 3x)$$

**P396.17.4**

$$A = \begin{pmatrix} -1 & -3 & 3 & -3 \\ -3 & -1 & -3 & 3 \\ 3 & -3 & -1 & -3 \\ -3 & 3 & -3 & -1 \end{pmatrix} \quad A4E = \begin{pmatrix} -3 & -3 & 3 & -3 \\ -3 & 3 & -3 & 3 \\ 3 & -3 & 3 & -3 \\ -3 & 3 & -3 & 3 \end{pmatrix}$$

$\therefore$  秩  $(A+4E)=1 \quad \therefore \lambda_1 = 4$  至少为 3 重根, 而

$$-(4+4+4) + \lambda_2 = \text{Tr}(A) = -4. \Rightarrow \lambda_2 = 8$$

$\lambda_1 = -4$  解  $(A+4E)x=0$ , 即  $x_1 - x_2 + x_3 - x_4 = 0$

得正交基础体系

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}$$

单位化为

$$\frac{1}{12} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{12}} \begin{pmatrix} 1 \\ -1 \\ - \\ 3 \end{pmatrix}$$

$\lambda_2$  8解  $(A-8E)x=0$ . 得解取自  $A+4E$  的一列

$$\begin{pmatrix} 3 \\ -3 \\ 3 \\ -3 \end{pmatrix}$$

单位化为

$$\frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\text{令 } T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{12} & -1/2 \\ 0 & -2/\sqrt{6} & 1/\sqrt{12} & 1/2 \\ 0 & 0 & 3/\sqrt{12} & -1/2 \end{pmatrix}$$

$$\text{则 } T'AT = T^{-1}AT = \begin{pmatrix} -4 & & & \\ & -4 & & \\ & & -4 & \\ & & & 8 \end{pmatrix}$$

P395.10.1  $0 \in V_1 \neq \emptyset$

$$\forall \beta_1, \beta_1 \in V_1 \quad \left. \begin{aligned} (\beta_1, \beta_2, \alpha) = (\beta_1, \alpha) + (\beta_2, \alpha) = 0 &\Rightarrow \beta_1 + \beta_2 \in V_1 \\ (k\beta_1, \alpha) = k(\beta_1, \alpha) = 0 &\Rightarrow k\beta_1 \in V \end{aligned} \right\} \therefore V_1 \leq V.$$

P395.10.2  $\because \alpha \neq 0 \quad \therefore \alpha \notin V_1 \quad \text{故 } \dim V_1 \leq n-1.$

将  $\alpha$  扩充为  $V$  的一个正交基  $\alpha_1 = \alpha, \alpha_2, \alpha_3 \cdots \alpha_n$ , 那么.

$$\alpha_i \in V_1 (i \geq 2) \quad \therefore L(\alpha_2, \alpha_3 \cdots \alpha_n) \leq V_1 \Rightarrow \dim V_1 \geq n-1.$$

$$\therefore \dim V_1 \geq n-1.$$

**P394, 11①** 设两个基:  $\varepsilon_1, \varepsilon_2, \cdots \varepsilon_n$  及  $\eta_1, \eta_2, \cdots \eta_n$ , 它们的度量矩阵分别为  $A$  和  $B$ , 并设

$$(\eta_1, \eta_2, \dots, \eta_n) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)C$$

$$\text{任设 } \alpha, \beta \in V, \alpha = (\varepsilon_1, \dots, \varepsilon_n)X_1 = (\eta_1, \eta_2, \dots, \eta_n)X_2$$

$$\beta = (\varepsilon_1, \dots, \varepsilon_n)Y_1 = (\eta_1, \eta_2, \dots, \eta_n)Y_2$$

$$\text{所以 } X_1 = CX_2, Y_1 = CY_2$$

$$(\alpha, \beta) = X_2'BY_2 = X_1'AY_1 = X_2'(C'AC)Y_2$$

$$\therefore C'AC = B(\text{合同})$$

**P394.11②,**

取V的一个基  $\alpha_1, \alpha_2, \dots, \alpha_n$ , 其度量矩阵为A, 因为A正交, 故存在矩阵C, 使

$$C'AC = E$$

做基  $(\eta_1, \eta_2, \dots, \eta_n) = (\alpha_1, \alpha_2, \dots, \alpha_n)C$ , 那么  $\eta_1, \eta_2, \dots, \eta_n$  的度量矩阵为  $C'AC = E$ ,

因此  $\eta_1, \eta_2, \dots, \eta_n$  为标准正交基.

**P394.12,**  $\alpha_1, \alpha_2, \dots, \alpha_m \in V$   $\alpha_{ij} = (\alpha_i, \alpha_j)$  记:

$$G(\alpha_1, \alpha_2, \dots, \alpha_m) = (\alpha_{ij})_{m \times m}$$

称  $G(\alpha_1, \alpha_2, \dots, \alpha_m)$  为  $\alpha_1, \alpha_2, \dots, \alpha_m$  的Gram矩阵

称  $|G(\alpha_1, \alpha_2, \dots, \alpha_m)|$  为  $\alpha_1, \alpha_2, \dots, \alpha_m$  的Gram行列式

证明  $\alpha_1, \alpha_2, \dots, \alpha_m$  线性无关  $\Leftrightarrow |G(\alpha_1, \alpha_2, \dots, \alpha_m)| \neq 0$

证: 若  $m=1$ ,  $\alpha_1$  线性无关  $\Leftrightarrow (\alpha_1, \alpha_1) > 0 \Leftrightarrow |G(\alpha_1)| = |\alpha_1|^2 \neq 0$ , 成立

若  $m > 1$ , 而  $|G(\alpha_1, \alpha_2, \dots, \alpha_m)| = 0$

不妨设  $A = (\beta_1, \beta_2, \dots, \beta_m)$

$$\Leftrightarrow \beta_j = \sum_{\substack{k=1 \\ k \neq j}}^m c_k \beta_k \Leftrightarrow \alpha_{ij} = \sum_{k \neq j} c_k \alpha_{ik} = \sum_{k \neq j} c_k (\alpha_i, \alpha_k)$$

$$\Leftrightarrow (\alpha_i, \alpha_j - \sum_{k \neq j} c_k \alpha_k) = 0, \therefore \Leftrightarrow \gamma = 0, i = 1, 2, \dots, m.$$

$$\because \gamma = \alpha_j - \sum_{k \neq j} c_k \alpha_k \in L(\alpha_1, \alpha_2, \dots, \alpha_m),$$

$$\Leftrightarrow \alpha_j = \sum_{k \neq j} c_k \alpha_k \Leftrightarrow \alpha_1, \alpha_2, \dots, \alpha_m \text{ 线性相关}$$

$$|G(\alpha_1)| = |\alpha_1|^2$$

$$|G(\alpha_1, \alpha_2)| = \left| \begin{pmatrix} (\alpha_1, \alpha_2), (\alpha_1, \alpha_2) \\ (\alpha_2, \alpha_1), (\alpha_2, \alpha_2) \end{pmatrix} \right| = \left| \begin{pmatrix} |\alpha_1|^2 & |\alpha_2| |\alpha_1| \cos \theta \\ |\alpha_1| |\alpha_2| \cos \theta & |\alpha_2|^2 \end{pmatrix} \right|$$

$$= |\alpha_1|^2 |\alpha_2|^2 (1 - \cos^2 \theta) = (|\alpha_1| |\alpha_2| \cos \theta)^2$$

类似地:

$$|G(\alpha_1, \alpha_2, \alpha_3)| = (\text{平行六面体体积})^2$$

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ 0 & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{nn} \end{pmatrix}$$

**P394, 13, 设:**

因为A正交, 故A'A=E, 令A=(\beta\_1, \beta\_2, \cdots, \beta\_n)

由第1行列, \alpha\_{11}^2 = 1, \alpha\_{11} = \pm 1

由\beta\_1与其余各列正交, \beta\_1 \perp \beta\_j (j>1), (\beta\_1, \beta\_j) = \alpha\_{11}\alpha\_{1j} = 0 \Rightarrow \alpha\_{1j} = 0 (j>1)

$$\therefore A = \begin{pmatrix} \pm 1 & 0 \\ 0 & A_1 \end{pmatrix}$$

其中A\_1仍为上三角正交矩阵, 但阶数少1, 故可用归纳法给出证明, 且n=1时显然为真, 由归纳法原理, 证毕。

**P394, 14①, 设** A=(\alpha\_1, \alpha\_2, \cdots, \alpha\_n), 则\alpha\_1, \alpha\_2, \cdots, \alpha\_n做成\mathbb{R}^n的一个基, 用Schmidt方法把它们正交化\varepsilon\_1, \varepsilon\_2, \cdots, \varepsilon\_n, 由定理2(P9, 130, 4.1)

$$L(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_i) = L(\alpha_1, \alpha_2, \cdots, \alpha_i), \forall_i$$

$$\therefore (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n) \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ & t_{22} & \cdots & t_{2n} \\ & & \ddots & t_{n-1n} \\ & & & t_{nn} \end{pmatrix}, t_{ii} > 0$$

$$\text{令 } Q = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \text{ 正交, } T_1 = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ & \ddots & \\ 0 & & t_{nn} \end{pmatrix}, T = T_1^{-1}$$

$$\therefore Q = AT_1$$

$$A = QT_1^{-1} = QT$$

(唯一性)若A=QT=Q\_2T\_2

$$\therefore Q_2^{-1}Q = T_2T^{-1}$$

$\therefore$  上三角矩阵  $T_2T$  为正交矩阵  $Q_2^{-1}Q$

$\therefore T_2, T$  的对角线皆大于 0,  $\therefore T_2T^{-1}$  的对角线皆大于 0, 由 13 题 (见 P19, 138, 10. 1)

$$T_2T^{-1}=E, \therefore T_2=T, \text{ 满秩}$$

$$\therefore Q_2=Q$$

**P394, 14②**,  $\therefore A$  正交, 则存在  $C$  可逆使

$$A=C'C$$

而  $C$  可逆, 由 ①, 有  $C=QT$ ,  $Q$  正交  $T$  上三角。

$$\therefore A=C'C=T'Q'QT=T'ET=T'T$$

**P395, 15①**,  $A\alpha = \alpha - 2(\eta, \alpha)\eta$   $\eta \in V$  为一单位向量

$$\therefore (A\alpha, A\beta) = (\alpha - 2(\eta, \alpha)\eta, \beta - 2(\eta, \beta)\eta)$$

$$= (\alpha, \beta) - 2(\eta, \alpha)(\eta, \beta) - 2(\eta, \beta)(\alpha, \eta) + 4(\eta, \alpha)(\eta, \beta)(\eta, \eta)$$

$$= (\alpha, \beta) \quad \therefore A \text{ 保持内积}$$

$$\text{又 } A(k\alpha + l\beta) = k\alpha + l\beta - (2\eta, k\alpha + l\beta)\eta$$

$$= k(\alpha - 2(\eta, \alpha)\eta) + l(\beta - 2(\eta, \beta)\eta) = kA\alpha + lA\beta$$

$\therefore A$  为线性的, 故  $A$  是正交变换

**P395, 15②**, 以  $\eta$  为起点, 扩充一个标准正交基,  $\varepsilon_1 = \eta, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_n$

$$\text{则 } A\varepsilon_1 = \varepsilon_1 - 2\varepsilon_1 = -\varepsilon_1 \quad A\varepsilon_i = \varepsilon_i - 0 = \varepsilon_i (i \geq 2)$$

$$\therefore A(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}$$

$\therefore |A| = -1$ , 为第二类的

**P395, 15③**,  $\therefore 1$  至少为  $A$  的  $n-1$  重特征值, 特征子空间  $V, \dim V_1 = n-1$ ,

取  $V_1$  的标准正交基  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$  扩充为  $V$  的标准正交基  $\varepsilon_1, \dots, \varepsilon_{n-1}, \varepsilon_n$

$\therefore A\varepsilon_i = \varepsilon_i (i=1, 2, \dots, n-1)$ , 而  $A\varepsilon_n$  与  $A\varepsilon_i$  正交 ( $i=1, 2, \dots, n-1$ )

$$\therefore A\varepsilon_n \in (L(A\varepsilon_1, A\varepsilon_2, \dots, A\varepsilon_{n-1}))^\perp = (L(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}))^\perp = L(\varepsilon_n)$$

$A\varepsilon_n = \pm \varepsilon_n$  ( $\therefore A$  正交), 若  $A\varepsilon_n = \varepsilon_n \Rightarrow \dim V_1 = n$  矛盾,  $\therefore A\varepsilon_n = -\varepsilon_n$

$$\forall \alpha \in V, \quad \alpha = \sum_{i=1}^n x_i \varepsilon_i$$

$$A\alpha = \sum_{i=1}^{n-1} x_i \varepsilon_i - x_n \varepsilon_n = \alpha - 2x_n \varepsilon_n = \alpha - 2(\varepsilon_n, \alpha)\varepsilon_n$$

是一个镜面反射

P395,16, 若  $A' = -1$ ,  $\lambda_0$  为  $A$  的特征值,  $X_0$  为其特征向量

$$X_0 \neq 0, \quad \therefore \overline{X_0}' X_0 \neq 0$$

$$AX_0 = \lambda_0 X_0 \quad A\overline{X_0} = \overline{AX_0} = \overline{\lambda_0 X_0} = \overline{\lambda_0} \overline{X_0}$$

$$\therefore \lambda_0 \overline{X_0}' X_0 = \overline{X_0}' (\lambda_0 X_0) = \overline{X_0}' (AX_0) = -(\overline{X_0}' A') X_0 = -(\overline{AX_0})' X_0$$

$$= -(\overline{AX_0})' X_0 = -(\overline{\lambda_0 X_0})' X_0 = -\overline{\lambda_0} (\overline{X_0}' X_0)$$

$$\therefore (\text{由 } \overline{X_0}' X_0 \neq 0) \quad \lambda_0 = -\overline{\lambda_0} \quad \lambda_0 = 0 \text{ 或纯虚数}$$

$$|\lambda E - A| = \begin{vmatrix} \lambda - 2 & 2 & 0 \\ 2 & \lambda - 1 & 2 \\ 0 & 2 & \lambda \end{vmatrix} = \lambda^3 - 3\lambda^2 - 6\lambda + 8 = (\lambda - 1)(\lambda - 4)(\lambda + 2)$$

**P395, 17①:**

$$\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = -2$$

解:  $(A - \lambda_1 E)X = 0$

$$(A - E)X = 0, \quad X = (2, 1, -2)', \quad \varepsilon_1 = \frac{1}{3}(2, 1, -2)'$$

$$\text{解: } (A - 4E)X = 0, X = (2, -2, 1)', \quad \varepsilon_2 = \frac{1}{3}(2, -2, 1)'$$

$$\text{解: } (A + 2E)X = 0, X = (1, 2, 2)', \quad \varepsilon_3 = \frac{1}{3}(1, 2, 2)'$$

$$\text{令 } T = \frac{1}{3} \begin{pmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ -2 & 1 & 2 \end{pmatrix}$$

$$\text{则: } T'AT = T^{-1}AT = \begin{pmatrix} 1 & & \\ & 4 & \\ & & -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$$

**P395, 17②,**

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$$

解: ①:  $\because A - \lambda E$ , ( $\lambda$  用数 1 代),  $A - E$  的秩为 1

$$\therefore |\lambda E - A| = (\lambda - 1)^2 (\lambda - \lambda_2), \therefore \lambda_2 = \text{Tr}(A) - \lambda_1 - \lambda_1 = 10$$

$$2^\circ, (A - E)(A - 10E) = 0$$

$$\lambda_1 = 1, \text{解: } x_1 + 2x_2 - 2x_3 = 0, \text{得: } \eta_1 = \begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} 1 \\ 2 \\ 5/2 \end{pmatrix}$$

$$\varepsilon_1 = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{pmatrix}$$

$$\lambda_2 = 10, \text{取} A - E \text{的一列: } \eta_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad \varepsilon_2 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$$3^\circ, \text{取} T = \begin{pmatrix} 2/\sqrt{5} & 2/\sqrt{45} & 1/3 \\ -1/\sqrt{5} & 4/\sqrt{45} & 2/3 \\ 0 & 5/\sqrt{45} & -2/3 \end{pmatrix}$$

$$\text{则: } T'AT = T^{-1}AT = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \\ 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{pmatrix}$$

**P395,17③,**

$$\text{解: } |A - \lambda E| = \begin{vmatrix} -\lambda & 0 & 4 & 1 \\ 0 & -\lambda & 1 & 4 \\ 4 & 1 & -\lambda & 0 \\ 1 & 4 & 0 & -\lambda \end{vmatrix} = (5 - \lambda) \begin{vmatrix} 1 & 0 & 4 & 1 \\ 1 & -\lambda & 1 & 4 \\ 1 & 1 & -\lambda & 0 \\ 1 & 4 & 0 & -\lambda \end{vmatrix} = (5 - \lambda)(3 - \lambda) \begin{vmatrix} 1 & 0 & 4 & 1 \\ 1 & -\lambda & 1 & 4 \\ 1 & 1 & -\lambda & 0 \\ 0 & 1 & -1 & 1 \end{vmatrix} -$$

$$= (\lambda - 3)(\lambda - 5) \begin{vmatrix} 1 & 0 & 4 & 1 \\ 0 & -\lambda & 1 & 4 \\ 4 & 1 & -\lambda & 0 \\ 1 & 4 & 0 & -\lambda \end{vmatrix} = (\lambda - 3)(\lambda - 5) \begin{vmatrix} -\lambda & -3 & 3 \\ 1 & \lambda - 4 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= (\lambda - 3)(\lambda - 5) \begin{vmatrix} -\lambda & -3 - \lambda & 3 + \lambda \\ 1 & -\lambda - 3 & -1 - 1 \\ 1 & 0 & 0 \end{vmatrix} = (\lambda - 3)(\lambda - 5)(\lambda + 3) \begin{vmatrix} -1 & 1 \\ -\lambda - 3 & -2 \end{vmatrix}$$

$$= (\lambda - 3)(\lambda + 3)(\lambda - 5)(\lambda + 5)$$

$$\lambda_1 = 3, \lambda_2 = -3, \lambda_3 = 5, \lambda_4 = -5$$

解:  $(A - \lambda_1 E)X = 0$

$$(A - 3E)X = 0, \text{ 得: } X = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \varepsilon_1 = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)'$$

解:  $(A + 3E)X = 0, \text{ 得: } X = (1, -1, -1, 1)', \therefore \varepsilon_2 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)',$

解:  $(A - 5E)X = 0, \text{ 得: } X = (1, 1, 1, 1)', \therefore \varepsilon_3 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)',$

解:  $(A + 5E)X = 0, \text{ 得: } X = (1, 1, -1, -1)', \therefore \varepsilon_4 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)',$

$$\text{令: } T = \frac{1}{2} \begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$$

则:  $T'AT = T^{-1}AT = \begin{pmatrix} 3 & & & \\ & -3 & & \\ & & 5 & \\ & & & -5 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

**P395, 17⑤:**

解: 秩  $(A) = 1, \therefore \lambda_1 = 0$  为  $A$  的 3 重根 (特征根)

$$\lambda_1 + \lambda_1 + \lambda_1 + \lambda_2 = \text{Tr}(A), \therefore \lambda_2 = 4$$

$$\therefore A(A - 4E) = 0$$

对于  $\lambda_1 = 0$ , 解:  $x_1 + x_2 + x_3 + x_4 = 0$

得:

$$\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix} \text{ 单位化: } \varepsilon_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \\ 0 \end{pmatrix}, \varepsilon_3 = \begin{pmatrix} 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ -3/\sqrt{12} \end{pmatrix}$$



对于 $\lambda_2=4$ ,其解为 $A$ 的一列: $\alpha_4=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \varepsilon_4=\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$

令 $T=\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/2 \\ 0 & -2/\sqrt{6} & 1/\sqrt{12} & 1/2 \\ 0 & 0 & -3/\sqrt{12} & 1/2 \end{pmatrix}$

则 $T^{-1}AT=T'AT=\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 4 \end{pmatrix}$

$f=X'\begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}X, B\begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$

**P395, 18**① $f=X'$

$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, (H^{-1}=H'=H)$

另解法, 作正交矩阵,  $H=\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$\therefore A=H'BH-E=\begin{Bmatrix} 3 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 1 \end{Bmatrix}E=\begin{Bmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{Bmatrix}$ 即为170中的 $A$ (见P9,139,11,4)

取 $T=\frac{1}{9}\begin{pmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ -2 & 1 & 2 \end{pmatrix}$ 正交, 则 $T'AT=\begin{pmatrix} 1 & & \\ & 4 & \\ & & -2 \end{pmatrix}=T'(H'BH-E)T$

$\therefore (HT)'B(HT)=\begin{pmatrix} 1 & & \\ & 4 & \\ & & -2 \end{pmatrix}+T'ET=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}, HT=\frac{1}{3}\begin{pmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

$$\begin{cases} X_1 = -\frac{2}{3}y_1 + \frac{1}{3}y_2 + \frac{2}{3}y_3 \\ X_2 = \frac{1}{3}y_1 - \frac{2}{3}y_2 + \frac{2}{3}y_3 \\ X_3 = \frac{2}{3}y_1 + \frac{2}{3}y_2 + \frac{1}{3}y_3 \end{cases}$$

即令  $f = 2y_1^2 + 5y_2^2 - y_3^2$

$$f = X' \begin{pmatrix} 1 & -2 & 2 \\ -2 & -2 & 4 \\ 2 & 4 & -2 \end{pmatrix} X, B = \begin{pmatrix} 1 & -2 & 2 \\ -2 & -2 & 4 \\ 2 & 4 & -2 \end{pmatrix}$$

**P395, 18②**

又  $\because A = -B + 3E$  即为 17② 中的  $A$  (见 P9, 138, 10, 3)

$$\text{取 } T = \begin{pmatrix} 2/\sqrt{5} & 2/\sqrt{45} & 1/3 \\ -1/\sqrt{5} & 4/\sqrt{45} & 2/3 \\ 0 & 5/\sqrt{45} & -2/3 \end{pmatrix}, \text{ 则 } T'AT = T^{-1}(3E - B)T = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$$

$$\therefore T^{-1}BT = T^{-1}(3E)T - \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix} = \begin{pmatrix} 2 & & \\ & 2 & \\ & & -7 \end{pmatrix}$$

$$\text{令 } \begin{cases} x_1 = \frac{2}{\sqrt{5}}y_1 + \frac{2}{\sqrt{45}}y_2 + \frac{1}{3}y_3 \\ x_2 = -\frac{1}{\sqrt{5}}y_1 + \frac{4}{\sqrt{45}}y_2 + \frac{2}{3}y_3 \\ x_3 = \frac{5}{\sqrt{45}}y_2 - \frac{2}{3}y_3 \end{cases}, \text{ 则 } f = 2y_1^2 + 2y_2^2 - 7y_3^2$$

$$f = X' \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} X, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

**P395, 18③,**

$$\text{解 } |\lambda E - B| = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1), \therefore \varepsilon_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

令  $T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ , 则  $T_1$  正交, 且

$$T_1'BT_1 = T_1^{-1}BT_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{令 } T = \begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix} \text{ 也正交, } \therefore T'AT = T^{-1}AT = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$\text{令 } \begin{cases} x_1 = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2 \\ x_2 = \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{2}}y_2 \\ x_3 = \frac{1}{\sqrt{2}}y_3 + \frac{1}{\sqrt{2}}y_4 \\ x_4 = \frac{1}{\sqrt{2}}y_3 - \frac{1}{\sqrt{2}}y_4 \end{cases}$$

$$\text{则 } f = y_1^2 - y_2^2 + y_3^2 - y_4^2$$

$$f = X' \begin{pmatrix} 1 & -1 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 3 & -2 & 1 & -1 \\ -2 & 3 & -1 & 1 \end{pmatrix} X = X'AX$$

**P395, 18④,**

$$\text{解: } |\lambda E - A| = \begin{vmatrix} \lambda-1 & 1 & -3 & 2 \\ 1 & \lambda-1 & 2 & -3 \\ -3 & 2 & \lambda-1 & 1 \\ -2 & -3 & 1 & \lambda-1 \end{vmatrix} = (\lambda-1) \begin{vmatrix} 1 & 1 & -3 & 2 \\ 1 & \lambda-1 & 2 & -3 \\ 1 & 2 & \lambda-1 & 1 \\ 1 & -3 & 1 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)(\lambda-7) \begin{vmatrix} 1 & 1 & -3 & -2 \\ 1 & \lambda-1 & 2 & -3 \\ 1 & 2 & \lambda-1 & 1 \\ 0 & 1 & -1 & 1 \end{vmatrix} = (\lambda-1)(\lambda-7) \begin{vmatrix} 1 & -2 & -3 & -1 \\ 1 & \lambda+1 & 2 & -1 \\ 1 & \lambda+1 & \lambda-1 & \lambda \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

$$= (\lambda-1)(\lambda-7)(-1)(-1)(\lambda+3) \begin{vmatrix} 1 & -2 & -1 \\ 1 & \lambda+1 & -1 \\ 1 & \lambda+1 & \lambda \end{vmatrix}$$

$$= (\lambda-1)(\lambda-7) \begin{vmatrix} 1 & -2 & -1 \\ 0 & \lambda+3 & 0 \\ 0 & \lambda+3 & \lambda+1 \end{vmatrix} = (\lambda-1)(\lambda-7)(\lambda+1)(\lambda+3)$$

$$\therefore \lambda_1 = 1, \lambda_2 = 7, \lambda_3 = -1, \lambda_4 = -3$$

$$(A-\lambda_1 E)X=(A-E)X=0, \text{ 得 } X=(1, 1, 1, 1)', \varepsilon_1=\frac{1}{2}(1,1,1,1)'$$

$$(A-\lambda_2 E)X=(A-7E)X=0, \text{ 得 } X=(1, -1, 1, -1)', \varepsilon_2=\frac{1}{2}(1,-1,1,-1)'$$

$$(A-\lambda_3 E)X=(A+E)X=0, \text{ 得 } X=(1, 1, -1, -1)', \varepsilon_3=\frac{1}{2}(1,1,-1,-1)'$$

$$(A-\lambda_4 E)X=(A+3E)X=0, \text{ 得 } X=(1, -1, -1, 1)', \varepsilon_4=\frac{1}{2}(1,-1,-1,1)'$$

$$\therefore \text{ 令 } T=\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \text{ 则 } T'AT=T^{-1}AT=\begin{pmatrix} 1 & & & \\ & 7 & & \\ & & -1 & \\ & & & -3 \end{pmatrix}$$

$$\begin{cases} X_1 \\ X_2 \\ X_3 \\ X_4 \end{cases} = T \begin{cases} y_1 \\ y_2 \\ y_3 \\ y_4 \end{cases} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{cases} y_1 \\ y_2 \\ y_3 \\ y_4 \end{cases}, \text{ 则}$$

$$f = y_1^2 + 7y_2^2 - y_3^2 - 3y_4^2$$

**P395,19**,  $\because A$  实对称, 存在正交矩阵  $T$ , 使

$$T'AT=T^{-1}AT=\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}=D$$

其中  $\lambda_1, \lambda_2, \dots, \lambda_n$  为  $A$  的所有特征根

$A$  正交  $\Leftrightarrow D$  正交  $\Leftrightarrow$  正惯性指数  $=n \Leftrightarrow \lambda_i > 0 (\forall i=1, 2, \dots, n)$

**P396.20** “充分性”, 设  $\lambda_1$  为  $A$  的实特征根, 取  $\lambda_1$  的单位特征向量  $\varepsilon_1$ , 扩充为  $\mathbb{R}^n$  的标准正交基  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ , 取  $T_1=(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  为正交矩阵

$$T_1^{-1}AT_1=T_1'AT_1=\begin{pmatrix} \lambda_1 & \alpha \\ & A_1 \end{pmatrix}$$

$\because |\lambda E - A| = (\lambda - \lambda_1) |\lambda E - A_1|$ , 故  $A_1$  的特征根全为实根, 且阶数少, 故由归纳假设 ( $n=1$ , 显然成立), 存在  $T_2$  正交。

$B_2 = T_2' A_1 T_2$  为上三角矩阵

作正交矩阵  $T, T_3$

$$T_3 = \begin{pmatrix} 1 & 0 \\ 0 & T_2 \end{pmatrix}, T = T_1 T_3$$

$$\text{那么, } T'AT = T^{-1}AT = T_3^{-1}(T_1^{-1}AT_1)T_3 = T_3^{-1} \begin{pmatrix} \lambda_1 & \alpha \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & T_2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & \alpha T_2 \\ 0 & T_2^{-1}A_1T_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \beta \\ 0 & B_1 \end{pmatrix}$$

为上三角矩阵

"必要性", 若  $TAT = T^{-1}AT = B$  为上三角

$$\because A, T \in \mathbb{R}^{n \times n}, \therefore B \in \mathbb{R}^{n \times n}$$

$$\because |\lambda E - A| = |\lambda E - B| = \prod_{i=1}^n (\lambda - b_{ii}) \text{ 全为实特征根 } b_{11}, b_{22}, \dots, b_{nn},$$

**P396, 21** “必要性”  $T^{-1}AT = B$ , 则  $A, B$  相似, 故特征值全部相同,

“充分性”, 若  $A, B$  的特征值都由  $\lambda_1, \lambda_2, \dots, \lambda_n$ ,

则存在,  $T_1, T_2$  正交, 使

$$T_1'AT_1 = T_1^{-1}AT_1 = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}, \quad T_2'BT_2 = T_2^{-1}BT_2 = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix} = D$$

$$\therefore T_1^{-1}AT_1 = T_2^{-1}BT_2$$

令  $T = T_1T_2^{-1} = T_1T_2'$  也是正交的, 且

$$T^{-1}AT = T_2(T_1^{-1}AT_1)T_2^{-1} = T_2DT_2^{-1} = B$$

$$\text{P396, 22, } A' = A, A^2 = A, \text{ 证存在 } T \text{ 正交, } T'AT = T^{-1}AT = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 0 \end{pmatrix}$$

证:  $\because A$  实对称, 故必有正交矩阵  $T$  使

$$T^{-1}AT = T'AT = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

(其中特征值  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$ )

$\because A(A - E) = 0$ , 最小多项式为  $x(x-1)$  的因式,  $\therefore$  特征多项式为  $(x-1)^r x^{n-r}$

而  $\lambda_i$  为特征值,  $X - \lambda_i \mid (x-1)^r x^{n-r} \Rightarrow \lambda_i = 1$  或  $0$

$$\therefore \lambda_1 = \lambda_2 = \dots = \lambda_r = 1, \lambda_{r+1} = \dots = \lambda_n = 0$$

证毕.

P396,23  $A \in L(V)$  为正交变换, 子空间  $W \leq V$  为  $A$  的不变子空间

$\forall \alpha \in W^\perp$ , 由于对  $\forall \beta \in W$  有  $A\beta \in W$  (必须假设  $\dim W$  有限)

$\because \dim W$  有限,  $\therefore AW=W, \forall \gamma \in W$ , 必有  $\beta \in W$ , 使  $A\beta=\gamma$

$$\therefore (A\alpha, \gamma) = (A\alpha, A\beta) = (\alpha, \beta) = 0$$

因此,  $A\alpha \in W^\perp$

故  $W^\perp$  也是  $A$  的不变子空间

(注: 若  $\dim W = \infty, V = \{\alpha = (x_1, x_2, \dots, x_n, \dots) \mid x_i \text{ 中有限个非0}\}, W = \{\alpha \in V \mid x_1 = \dots = x_r = 0\}$

$A\alpha = (0, x_1, x_2, x_3, \dots, x_n, \dots)$  内积为对应分量之积之和, 则  $A$  为正交变换.

$W$  为  $A$ -子空间,  $W^\perp = \{\alpha \in V \mid x_{r+1} = x_{r+2} = \dots = x_n = \dots = 0\}$ , 不是  $A$ -子空间

$\because \gamma = (0, 0, \dots, \overset{(r \uparrow)}{1}, 0, \dots) \in W^\perp$ , 但  $A\gamma = (0, \dots, \overset{(r+1 \uparrow)}{0}, 1, 0, \dots, 0) \notin W^\perp$ )

P396、24① “必要性”, 若  $A$  反对称, 在标准正交基  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$  下

$$A(\varepsilon_1, \dots, \varepsilon_n) = (\varepsilon_1, \dots, \varepsilon_n)A \quad A = (a_{ij})_{n \times n}$$

$$\text{则: } a_{ij} = (A\varepsilon_j, \varepsilon_i) = -(\varepsilon_j, A\varepsilon_i) = -(A\varepsilon_i, \varepsilon_j) = -a_{ji}.$$

$$\therefore A' = -A$$

$$\text{“充分性”, 若 } A(\varepsilon_1, \dots, \varepsilon_n) = (\varepsilon_1, \dots, \varepsilon_n)A \quad \therefore A' = -A$$

$$\forall \alpha = (\varepsilon_1, \dots, \varepsilon_n)X \quad \beta = (\varepsilon_1, \dots, \varepsilon_n)Y. \quad A\alpha, A\beta \text{ 以坐标为 } AX, AY.$$

$$\therefore (A\alpha, \beta) = (AX)'Y = X'A'Y = -X'AY = -X'(AY) = -(\alpha, A\beta)$$

即,  $A$  为反对称的

P396、242 设  $V_1$  为  $A$ -子空间,  $A$  反对称。

$$W = V^\perp \quad \forall \alpha \in W \quad \forall \beta \in V_1 \quad \therefore {}_A \beta \in V_1$$

$$\therefore (A\alpha, \beta) = (\alpha, {}_A \beta) = 0$$

$$\therefore A\alpha \in W \quad \text{故 } W \text{ 为 } A\text{-子空间。}$$

P397.25. 设  $V = V_1 \oplus V_1^\perp$

必要性, 若  $\alpha = \beta + \gamma, (\beta \in V_1, \gamma \in V_1^\perp)$ , 则  $\forall \xi \in V_1, \alpha - \beta \perp \beta - \xi$

$$\therefore |\alpha - \xi|^2 = |(\alpha - \beta) + (\beta - \xi)|^2 = |\alpha - \beta|^2 + |\beta - \xi|^2 \geq |\alpha - \beta|^2$$

$$\text{即 } |\alpha - \beta| \leq |\alpha - \xi|.$$

$$\alpha \text{ 在 } V_1 \text{ 的分解式, } \alpha = \alpha_1 + \alpha_2 \quad \alpha_1 \in V_1 \quad \alpha_2 \in V_1^\perp$$

$$\text{于是 } |\alpha - \alpha_1| \leq |\alpha - \beta| \leq |\alpha - \alpha_1| \quad \text{又 } \because \beta - \alpha_1 \in V_1$$

充分性, 取  $\therefore |\alpha - \alpha_1|^2 = |\alpha - \beta|^2 + |\beta - \alpha_1|^2 \Rightarrow |\beta - \alpha_1| = 0 \Rightarrow \beta = \alpha_1$  必为内射影。

P396,26, 证 1)  $(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp$  和 2)  $(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp$

$$\text{证 } 1) \because (V_1 + V_2)^\perp \subseteq V_1^\perp, V_2^\perp \quad \therefore (V_1 + V_2)^\perp \subseteq V_1^\perp \cap V_2^\perp$$

$$\text{反过来, } \forall \alpha \in V_1^\perp \cap V_2^\perp, \quad \text{则} \forall \beta \in V_1 + V_2 \quad \beta = \beta_1 + \beta_2 (\beta_i \in V_i)$$

$$\because \alpha \perp \beta_1, \alpha \perp \beta_2, \therefore \alpha \perp \beta_1 + \beta_2 = \beta \Rightarrow \alpha \in (V_1 + V_2)^\perp$$

$$\therefore V_1^\perp \cap V_2^\perp \subseteq (V_1 + V_2)^\perp \quad \text{故} (V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp$$

2), 由于正交补是唯一的

$$\therefore (V_1 \cap V_2)^\perp = ((V_1^\perp)^\perp \cap (V_2^\perp)^\perp)^\perp = ((V_1^\perp + V_2^\perp)^\perp)^\perp = V_1^\perp + V_2^\perp$$

$$P396, 27 \quad A = \begin{pmatrix} 0.39 & -1.89 \\ 0.61 & -1.80 \\ 0.93 & -1.68 \\ 1.35 & -1.50 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A'A = \begin{pmatrix} 3.2116 & -5.4225 \\ -5.4225 & 11.8845 \end{pmatrix} \quad A'B = \begin{pmatrix} 3.28 \\ -6.87 \end{pmatrix}$$

$$\therefore A'AX = A'B, \quad |A'A| = 8.76475395 = d$$

$$\text{得: } d_x = 1.728585, d_y = -4.277892$$

$$\therefore X = \frac{d_x}{d} = 0.197220025 \approx 0.197 \quad M = \frac{d_y}{d} = -0.48867896 \approx -0.488$$

设  $A = (\alpha_1, \alpha_2)$ , 故  $B$  到子空间  $W = L(\alpha_1, \alpha_2)$  的垂足为  $0.197\alpha_1 - 0.488\alpha_2$

$B$  到  $W$  的距离为  $|B - 0.197\alpha_1 + 0.488\alpha_2|$

$$= \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0.07683 \\ 0.12017 \\ 0.18321 \\ 0.26595 \end{pmatrix} - \begin{pmatrix} 0.92232 \\ 0.8784 \\ 0.81984 \\ 0.732 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0.00085 \\ 0.00143 \\ -0.00305 \\ 0.00205 \end{pmatrix} \right\| = \sqrt{0.000016272} = 0.004033906 \approx 0.00403$$

**P397 补 1**, 设  $\lambda$  为  $A$  (正交) 的特征值, 定义  $AX = \lambda X$ , 则  $A$  为正交变换

$$\because AX_0 = \lambda X_0 \Rightarrow |AX_0| = |X_0| \Rightarrow |\lambda| = 1 \Rightarrow \lambda = \pm 1, (\because \lambda \in \mathbb{R})$$

**P397, 补 2**,  $A$  为  $V$  中正交变换,  $|A| = 1$

设  $A$  的特征值为  $\lambda_1, \lambda_2, \dots, \lambda_n, \because |\lambda_i| = 1$ , 及  $f_A(x)$  是一个实系数多项式,

$\therefore \lambda_1, \lambda_2, \dots, \lambda_n$  中有  $k$  对共轭之积为 1, 剩下的  $n-2k$  个实根  $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_{n-2k}}$  之积为 1。

$\because \lambda_{i_s} = \pm 1, \therefore -1$  的个数必为偶数个, 而  $\dim V$  为奇数, 因此至少有一个特征值为实数 1

**P397 补 3** (仿上题),  $\because |A| = -1$ , 剩下的  $n-2k$  个实根之积为  $-1$ , 其中必有特征值  $= -1$ 。

**P397, 补 4**: 令  $r = A(k\alpha + l\beta - kA\alpha - lA\beta), \quad \because A$  得内积

$$\begin{aligned}
& \therefore (\gamma, \gamma) = A(k\alpha + l\beta), A(k\alpha + l\beta)) - (A(k\alpha + l\beta), kA\alpha) - (A(k\alpha + l\beta), lA\beta) \\
& \quad - (kA\alpha, A(k\alpha + l\beta)) + (kA\alpha, kA\alpha) + (k(A\alpha, lA\beta) - (lA\beta, A(k\alpha + l\beta))) \\
& \quad + (lA\beta, kA\alpha) + (lA\beta, lA\beta) \\
& = (k\alpha + l\beta, k\alpha + l\beta) - k(k\alpha + l\beta, \alpha) - l(k\alpha + l\beta, \beta) - k(\alpha, k\alpha + l\beta) + k^2(\alpha, \alpha) + kl(\alpha, \beta) \\
& \quad - l(\beta, k\alpha + l\beta) + kl(\beta, \alpha) + l^2(\beta, \beta) \\
& = (k\alpha + l\beta - k\alpha - l\beta, k\alpha + l\beta - k\alpha - l\beta) = 0, \therefore r = 0 \\
& \text{而 } \forall k, l, \alpha, \beta, A(k\alpha + l\beta) = kA\alpha + lA\beta, \therefore A \in L(V), \text{故 } A \text{ 是正交变换}
\end{aligned}$$

$$P397 \text{补}5, "必要性": (\beta_i, \beta_j) = (A\alpha_i, A\alpha_j) = (\alpha_i, \alpha_j)$$

$$"充分性"(归纳法) m=1 \text{ 时, } |\alpha_1| = |\beta_1|$$

$$\text{作标准正交基 } \varepsilon_1 = \frac{1}{|\alpha_1|} \alpha_1, \varepsilon_2, \dots, \varepsilon_n \text{ 及 } \eta_1 = \frac{1}{|\beta_1|} \beta_1, \eta_2, \dots, \eta_n, \text{ 则线性变换 } A: \varepsilon_i \rightarrow \eta_i$$

是正交变换, 则  $A\alpha_1 = A|\alpha_1| \varepsilon_1 = |\beta_1| \eta_1 = \beta_1$ , 而为所求设  $m-1$  成立, 考虑  $m$  情形

由假设有正交变换  $A_1, \alpha_i \rightarrow \beta_i, \alpha_m \rightarrow \tilde{\beta}_m, i=1, 2, \dots, m-1$ , 由于  $A_1$  保持内积及 Gram 矩阵, 行列式的线性相关系。

$$\beta_1, \dots, \beta_{m-1}, \tilde{\beta}_m \text{ 与 } \beta_1, \beta_2, \dots, \beta_{m-1}, \beta_m$$

$$\text{任何一个局部的线性关系相同, 设 } W = L(\beta_1, \dots, \beta_{m-1})$$

$$V_1 = L(\beta_1, \dots, \beta_{m-1}, \tilde{\beta}_m) = W + L(\tilde{\beta}_m), V_2 = L(\beta_1, \dots, \beta_m) = W + L(\beta_m)$$

$$(1) \text{ 若 } \tilde{\beta}_m \in W, \text{ 则 } \beta_m \in W, \text{ 设 } W \text{ 的标准正交基 } \varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$$

$$\therefore (\tilde{\beta}_m, \beta_i) = (\beta_m, \beta_i) \Rightarrow \forall i (\tilde{\beta}_m, \varepsilon_i) = (\beta_m, \varepsilon_i) \Rightarrow \tilde{\beta}_m = \beta_m, \text{ 则 } A, \text{ 即已为所求}$$

$$(2) \text{ 若 } \tilde{\beta}_m \notin W, \text{ 则 } \beta_m \notin W, \text{ 设 } V_1 \text{ 的标准正交基 } \varepsilon_1, \dots, \varepsilon_r, \tilde{\varepsilon}_{r+1}, V_2 \text{ 的标准正交基 } \varepsilon_1, \dots, \varepsilon_r, \varepsilon_{r+1}, \text{ 分别扩充为 } V \text{ 的标准正交基, } \dots, \tilde{\varepsilon}_{\gamma+1}, \dots, \tilde{\varepsilon}_n, \text{ 及 } \dots, \varepsilon_{\gamma+1}, \dots, \varepsilon_n, \text{ 作线性变换 } A,$$

$$\varepsilon_i \rightarrow \varepsilon_i, \tilde{\varepsilon}_j \rightarrow \begin{pmatrix} i \leq r \\ j > r \end{pmatrix}, \text{ 是一个正交变换, 而 } \beta_m(\tilde{\beta}_m) \text{ 用 } \varepsilon_1, \dots, \varepsilon_r, \varepsilon_{r+1} \text{ (或 } \tilde{\varepsilon}_{r+1}) \text{ 表示时的系数}$$

完全由  $\beta_i, \beta_j$  之间的内积确定, 由充分已知条件, 这些系数对应相等.

$$\therefore A_2 \tilde{\beta}_m = A_2(\alpha_1 \varepsilon_1 + \dots + \alpha_r \varepsilon_r + \alpha_{r+1} \tilde{\varepsilon}_{r+1}) = \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + \dots + \alpha_r \varepsilon_r + \alpha_{r+1} \varepsilon_{r+1} = \beta_m$$

$$\text{且 } A_2 \text{ 在 } W \text{ 上不动, 即, } A_2 \beta_i = \beta_i (1 \leq i \leq r)$$

$$\text{取 } A = A_2 A_1 \text{ 的合成, 则: } A: \alpha_i \rightarrow \beta_i \quad (i=1, 2, \dots, m)$$

且  $A$  为正交变换, 即为所求.

$$P397 \text{补}6, \therefore A \text{ 实对称, } \therefore \text{存在 } T \text{ 正交, 使}$$



$$T'AT = T^{-1}AT = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \text{其中 } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \text{为 } A \text{ 的特征值}$$

$\therefore A^2 = E, \therefore A$  的最小多项式为  $x^2 - 1$  的因式, 故

$A$  的特征多项式为  $(x-1)^r(x+1)^{n-r}$ , 即  $A$  的特征值为 1 或  $-1$

$\therefore$  当  $\lambda_i = \pm 1$ , 因此  $\lambda_1 = \cdots = \lambda_r = 1, \lambda_{r+1} = \cdots = \lambda_n = -1$

$$\text{即: } T^{-1}AT = \begin{pmatrix} E_r & 0 \\ 0 & -E_{n-r} \end{pmatrix}$$

**P397 补 7**, 作正交替换  $X=TY, \therefore X'X' = (Y'T')(TY) = Y'Y$

使  $f = X'AX = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2, \quad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$

$\therefore f = X'AX \leq \lambda_n (y_1^2 + y_2^2 + \cdots + y_n^2) = \lambda_n Y'Y = \lambda_n X'X$

$f = X'AX \geq \lambda_1 (y_1^2 + y_2^2 + \cdots + y_n^2) = \lambda_1 Y'Y = \lambda_1 X'X$ , 即得证

P397. 补8 设  $f = X'AX$ , 且正交替换  $X = TY$ , 使  $(\lambda_1 = \lambda)$

$$f = \lambda y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \quad \text{令 } Y_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \overline{X} = TY_0 = \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \\ \vdots \\ \overline{x_n} \end{pmatrix} \in R^n$$

$$\begin{aligned} \therefore f(\overline{X}) &= f(\overline{x_1}, \overline{x_2}, \cdots, \overline{x_n}) = \lambda = \lambda Y_0' Y_0 = \lambda (T' \overline{X})' (T' \overline{X}) \\ &= \lambda (\overline{X}' (TT') \overline{X}) = \lambda (\overline{X}' \overline{X}) = \lambda (\overline{x_1}^2 + \overline{x_2}^2 + \cdots + \overline{x_n}^2) \end{aligned}$$

**P397, 补 9①**, 取  $\eta = \alpha - \beta \neq 0, \eta_0 = \frac{1}{|\eta|} \eta$

作镜面反射,  $A: \xi \rightarrow \xi - 2(\eta_0' \xi) \eta_0, \forall \xi$

$$\text{则 } A\alpha = \alpha - 2 \left( \frac{1}{|\eta|} (\alpha - \beta), \alpha \right) \eta_0 = \alpha - 2 \left( \frac{1}{|\eta|} (\alpha - \beta), \alpha \right) \frac{1}{|\eta|} \eta$$

$$\therefore 2 \left( \frac{1}{|\eta|} (\alpha - \beta), \alpha \right) \frac{1}{|\eta|} = 2 \frac{(\alpha - \beta, \alpha)}{(\alpha - \beta, \alpha - \beta)} = 2 \frac{|\alpha|^2 - (\alpha, \beta)}{|\alpha|^2 - 2(\alpha, \beta) + |\beta|^2} = 2 \frac{1 - (\alpha, \beta)}{2 - 2(\alpha, \beta)} = 1$$

$\therefore A\alpha = \alpha - \eta = \beta$ , 即为所求.

**P397, 补 9②**, 设正交变换  $A$ , 标准正交基:  $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n \rightarrow \eta_1, \eta_2, \cdots, \eta_n$

作镜面反射  $B_1: \varepsilon_1 \rightarrow \eta_1, \varepsilon_i \rightarrow B_1 \varepsilon_i \quad (i > 1)$

$$\therefore L(B_1 \varepsilon_2, B_1 \varepsilon_3, \cdots, B_1 \varepsilon_n) = L(\varepsilon_1)^\perp \rightarrow L(\eta_1)^\perp = L(\eta_2, \eta_3, \cdots, \eta_n)$$

不妨设  $B_k$  是一系列镜面反射使:

$$\varepsilon_1 \rightarrow \eta_1, \cdots, \varepsilon_k \rightarrow \eta_k, \quad \varepsilon_{k+1} \rightarrow B_k \varepsilon_{k+1}, \cdots, \varepsilon_n \rightarrow B_k \varepsilon_n$$

作一镜面反射  $C_k: \xi_1 = B_k \varepsilon_{k+1} - \eta_{k+1} \quad \xi_0 = \frac{1}{|\xi_1|} \xi$

$\mathbb{C}_k: \alpha \rightarrow \alpha - 2(\xi_0, \alpha)\xi_0$  使  $B\varepsilon_{k+1} \rightarrow \eta_{k+1}$   
 $\because (\varepsilon_1, \dots, \varepsilon_k), \eta_1, \dots, \eta_k$  与  $\xi_1$  正交,  $\therefore l_{c_{kj}}\eta_i \rightarrow \eta_i (1 \leq i \leq k)$

令  $B_{k+1} = C_k B_k$  是一系列镜面反射之积, 且

$\varepsilon_1 \rightarrow \eta_1, \dots, \varepsilon_k \rightarrow \eta_k, \varepsilon_{k+1} \rightarrow \eta_{k+1}$   
 继续下去,  $n$  步后必存一系列反射之积  $B$  使

$$\varepsilon_1 \rightarrow \eta_1, \varepsilon_2 \rightarrow \eta_2, \dots, \varepsilon_n \rightarrow \eta_n$$

由线性变换的唯一存在性,  $A = B_n$  是一系列镜面反射之积

**P397, 补 10,** 设  $C$  可逆, 使  $C'BC = E$  ( $\because B > 0$ )

令  $A_1 = C'AC$ , 实对称, 存在正交  $Q$ , 使  $Q'A_1Q$  对角形

令  $T = CQ$ , 可逆, 则

$$T'AT = Q'(C'AC)Q = Q'A_1Q, \text{ 对角形}$$

$$T'BT = Q'(C'BC)Q = Q'EQ = E, \text{ 对角形, (证毕)}$$

**P398, 补 11,** 设:  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$  及  $\eta_1, \eta_2, \dots, \eta_n$  都为标准正交基, 解

$$(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)A = (\eta_1, \eta_2, \dots, \eta_n)$$

$\because \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$  及  $\eta_1, \eta_2, \dots, \eta_n$  的度量矩阵都是单位矩阵  $E$

任取  $\alpha, \beta \in \mathbb{C}^n, \alpha = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)X_1 = (\eta_1, \eta_2, \dots, \eta_n)X_2$

$$\beta = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)Y_1 = (\eta_1, \eta_2, \dots, \eta_n)Y_2$$

其中  $X_1 = AX_2, Y_1 = AY_2$

$$(\alpha, \beta) = \left( \sum x_i \varepsilon_i, \sum y_j \varepsilon_j \right) = \sum x_i \overline{y_j} (\varepsilon_i, \varepsilon_j) = X_1' \overline{Y_1} = X_2' A' \overline{AY_2}$$

$$= \left( \sum x_i \eta_i, \sum y_j \eta_j \right) = \sum x_i \overline{y_j} (\eta_i, \eta_j) = X_2' E \overline{Y_2}$$

由  $X_2, Y_2$  的任意性,  $A' \overline{A} = E$  故  $\overline{A}' A = E$ , 即  $A$  为酉矩阵

**P398 补 12,** 设  $A$  为酉矩阵,  $\lambda$  为其特征值,  $X_0 \neq 0, AX_0 = \lambda X_0$

$$\therefore \overline{AX_0} = \overline{AX_0} = \overline{\lambda X_0} \quad (\overline{AX_0})' = (\overline{\lambda X_0})' = \overline{\lambda}' \overline{X_0}'$$

$$\therefore |\lambda|^2 \overline{X_0}' X_0 = \overline{\lambda X_0}' \quad (\lambda X_0) = (\overline{AX_0})' (AX_0) = \overline{X_0}' (\overline{A}' A) X_0 = \overline{X_0}' X_0$$

$$\therefore \overline{X_0}' X_0 = |X_0|^2 \neq 0 \quad \therefore |\lambda|^2 = 1 \quad \text{即 } |\lambda| = 1$$

**P398, 补 13,** 设  $A$  复可逆,  $A = (\alpha_1, \alpha_2, \dots, \alpha_n)$ , 基  $\alpha_1, \alpha_2, \dots, \alpha_n$

用 Schmidt 方法, 将基正交化, 为  $\beta_1, \beta_2, \dots, \beta_n$ , 可知

$$(\beta_1, \beta_2, \dots, \beta_n) \begin{pmatrix} 1 & t_{12} & \cdots & t_{1n} \\ 0 & 1 & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = (\alpha_1, \alpha_2, \dots, \alpha_n) = (\beta_1, \beta_2, \dots, \beta_n) T_1$$

用将 $\beta_i$ 单位化,  $\gamma_i = \frac{1}{|\beta_i|} \beta_i$   $D = \begin{pmatrix} |\beta_1|^{-1} & & \\ & |\beta_2|^{-1} & \\ & & \ddots \\ & & & |\beta_n|^{-1} \end{pmatrix}$

可知标准正交基 $(\gamma_1, \gamma_2, \dots, \gamma_n) = (\beta_1, \beta_2, \dots, \beta_n)D$

令 $T_2 = T_1^{-1} = \begin{pmatrix} 1 & t_{12} & \cdots & t_{1n} \\ 0 & 1 & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$   $D^{-1} = \begin{pmatrix} |\beta_1| & & \\ & |\beta_2| & \\ & & \ddots \\ & & & |\beta_n| \end{pmatrix}$

$\therefore A = (\alpha_1, \alpha_2, \dots, \alpha_n) = (\beta_1, \beta_2, \dots, \beta_n)T_1 = (\gamma_1, \dots, \gamma_n)D^{-1}T_1$

即为上三角且对角线上全大于0, ( $t_{ii} = |\beta_i| > 0$ )

其次, 另设 $A = U_3 T_3$ , 一个分解, 则,  $UT = U_3 T_3$

$\tilde{U} = U_3^{-1}U = \bar{U}_3' U = T_3 T^{-1}$

为上三角的西矩阵 (类假正 13 题, P9, 138, 10.1 练习 13)

$\therefore \tilde{U}$ 为对角矩阵, 对角线大于0  $\Rightarrow \tilde{U} = E$

$\therefore T_3 = T, U_3 = U$ , (唯一性证毕)

**P398, 补 14,**  $\bar{A}' = A$ , 若 $\lambda_0$ 为特征值, 则有 $X_0 \neq 0, AX_0 = \lambda_0 X_0$

$\therefore \lambda_0 \bar{X}_0' X_0 = \bar{X}_0' (\lambda X_0) = \bar{X}_0' (AX_0) = (\bar{X}_0' A) X_0 = (\bar{X}_0' \bar{A}') X_0$   
 $= (\overline{AX_0})' X_0 = (\overline{\lambda_0 X_0})' X_0 = \overline{\lambda_0 X_0}' X_0 = \overline{\lambda_0} (\bar{X}_0' X_0)$

$\therefore \bar{X}_0' X_0 \neq 0 \quad \therefore \lambda_0 = \overline{\lambda_0} \quad \text{故 } \lambda_0 \in \mathbb{R}$

若A有两个特征值 $\lambda \neq \mu$ , 且 $X_1 \neq 0, X_2 \neq 0$ , 使 $AX_1 = \lambda X_1, AX_2 = \mu X_2$

$\therefore \lambda \bar{X}_2' X_1 = \bar{X}_2' (\lambda X_1) = \bar{X}_2' (AX_1) = (\bar{X}_2' A) X_1 = (\bar{X}_2' \bar{A}') X_1$   
 $= (\overline{AX_2})' X_1 = (\overline{\mu X_2})' X_1 = \overline{\mu X_2}' X_1 = \mu (\bar{X}_2' X_1)$

$\therefore \lambda \neq \mu \quad \therefore \bar{X}_2' X_1 = 0$

即 $(X_1, X_2) = X_1' \bar{X}_2 = (X_1' \bar{X}_2') = \bar{X}_2' X_1 = 0, \quad \therefore X_1 \perp X_2$